

The Australian National University

# Unions, Efficiency and Bargaining when Workers have Heterogeneous Productivity

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# Declaration

I certify that the thesis I have presented for examination for the PhD degree of The Australian National University is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

I declare that my thesis consists of 194 pages and approximately 40,000 words.

## **Statement of Conjoint Work**

I certify that this thesis was co-authored with Dr. Jose Rodrigues-Neto. Specifically, the modelling work was split evenly and undertaken in meetings over many years. The proof of uniqueness in Proposition 12 was prepared by Jose alone. The research topic, selection of methodology and connection with the literature (including the literature review) were entirely my work.

**Patrick Thomas Hamshere**

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# Abstract

This thesis proposes and analyzes micro-theoretic models of the strategic interaction between heterogeneous workers and a union, and that between a firm and a union. We assume union membership increases the bargaining power of the workers, but decreases their productivity. The workers and the firm produce a surplus. The union bargains on behalf of all workers, trying to maximize their surplus, which is a share of the total surplus. This thesis focuses on studying equilibria in which the least productive workers join the union, but the most productive workers do not. For each model, we show that such an equilibrium exists, is unique, and is robust to coalitional deviations. We find expressions for the equilibrium union size, union density, the wages of union and non-union workers, the surplus and profit of the firm, and the respective bargaining powers of the workers and the firm. The models' comparative statics are also studied. Equilibrium variables are compared across the models to gain insight into the union's preferences on performance-based pay. We also study how differences in the workers' productive levels affect their incentives to join the union. The most striking result that holds across the models is that the firm's surplus and profit can sometimes increase with the inefficiency coefficient of the union. This suggests there can exist situations where firms have a perverse incentive to make union workers less efficient compared to non-union workers. We also find that union membership and salaries decrease with the inefficiency coefficient of the union. This suggests that unionized workers always prefer to be as efficient as their non-unionized colleagues. We find that union wages

are generally lowest when they are linked to a worker's output, suggesting that unions may have incentives to oppose performance-based pay for their members.

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# Chapter 1

## Introduction

A common finding of researchers studying labor unions is that unions and unionization can affect a firm's efficiency, both positively and negatively, and increase the bargaining power of workers. For example, unions can help workers negotiate better salaries or employment conditions with their employer by representing them in negotiations for wages or enterprise agreements. The productivity of workers, and hence the profitability of the firm, can be adversely affected by union activities if, for example, they encourage restrictive work practices or promote inefficient hiring or firing. In contrast, union activities may improve productivity if they assist in the resolution of workplace issues.

An important issue in labor economics is therefore how a trade-off between efficiency and bargaining power affects the strategic interaction between workers, firms and unions. For example, if joining a labor union can improve the bargaining power of workers, but makes them less productive, then:

- How many workers can be expected to join the union?
- To what extent will a workforce be unionized?
- How does this level of unionization affect a firm's efficiency?

- How much effort should a firm make in bargaining with a union?
- How does the magnitude of the union efficiency affect the size of unions, the relative bargaining power of workers and the salaries of workers inside and outside unions?
- Who has incentives to join the union? Highly skilled workers might prefer not to join a union in the hope of obtaining a better job contract by negotiating individually with the employer.
- Under what conditions would there be full, or almost full, unionization of workers in a given industry?
- How does the distribution of worker skills and the bargaining power of the union affect salaries and the size of the union?
- What are the union's preferences on performance-based pay?

These are important questions because the rate of unionization in a workforce may have implications for economic efficiency, labor productivity, income equality, and labor laws. It may also raise political issues.

This thesis provides insight into the above questions by proposing and analyzing micro-theoretic models of the strategic interaction between heterogeneous workers and a union, and that between a firm and a union. It is assumed that joining the union can increase the bargaining power of workers, but affects their productivity. Because unions are essentially a coalition of workers, the strategic interaction between a union and coalitions of workers is also considered. Two main models are analyzed. The first is the Constant Wage Model of Chapter 3, where the strategic players are the union and the workers. The second is the Firm Choice Model of Chapter 6, where the strategic players are the union and the firm. Chapters 4, 5 and 7 examine variations of these baseline models by changing the assumptions on output observability and incorporating performance-based pay. They also compare the equilibrium variables across the models. Appendix B studies a variation of the baseline model that endogenizes the union efficiency effect.

The models assume a finite number of heterogeneous workers and each one decides whether to join the union. The heterogeneity of workers is in their skill; each worker has a different skill level. Workers and the firm produce a surplus and the union bargains on behalf of workers for their share of the surplus (which we call their wage). The union determines, and tries to maximize, wages for its members, subject to budget balance. Budget balance requires that union members share what is left from the workers' surplus after salaries to non-members are paid. Allowing for a large union decreases the total surplus, but, by assumption, increases the bargaining power of workers and hence the share of the surplus they receive. Chapters 6 and 7 allow for the firm to make costly effort in bargaining with the union.

This thesis focuses on studying equilibria in which the least productive workers join the union and the most productive workers do not. This thesis is concerned with studying the properties of such an equilibrium because, as described in Chapter 2, labor unions tend to be more popular amongst lower skilled workers (Blanchflower [5]). Across the models, we show that this equilibrium exists, is unique, and is robust to coalitional deviations. We find expressions for the equilibrium union size, union density (the share of unionized workers in the workforce), the wage of union and non-union workers, the surplus and profit of the firm and the bargaining power of workers and the firm. Comparative statics are also studied. Equilibrium variables are compared across the models to gain insight on the union's preferences on performance-based pay. We also study the incentives of workers with different productivity levels to join the union.

The most striking result that holds across all models is that the firm's surplus and profit can sometimes increase with the inefficiency coefficient. This suggests there can exist situations where firms have a perverse incentive to make union workers less efficient compared to non-union workers. The intuition is that the reduced efficiency causes unionized workers to exit the union and this increases the bargaining power of the firm and the total surplus. We find that union membership and salaries decrease with the

inefficiency coefficient of the union. In other words, the union and the workers always prefer to be as efficient as their non-unionized colleagues. We also find that union wages were generally lowest when they are linked to a worker's output. This suggests that unions may have incentives to oppose performance-based pay for their members.

This thesis derives results for populations of workers of any finite size, and computes the limits of all equilibrium variables, as the population of workers grows large without bound. The qualitative results regarding wages, how many workers will join a union and what types of workers have incentives to join, are shown to be consistent with results of empirical studies.

One limitation of models that study the strategic interaction between workers and a union is that they do not consider the supply and demand for labor or the action of the firm (Naylor [42]). This limitation also holds for the models in this thesis that study this strategic interaction (specifically, the Constant Wage Model of Chapter 3 and its variations in Chapters 4 and 5). This limitation is overcome to some degree by the models studied in Chapters 6 and 7, which focus on the strategic interaction between the union and the firm.

## 1.1 Contribution of this Thesis

The main contributions of this thesis are as follows. First, this thesis studies the strategic interaction between workers and a union when union membership increases the bargaining power of workers, but can affect their productivity. The novelty is that the strategic interaction is studied in the presence of union efficiency effects. Union efficiency effects have been described as an important area for further theoretical research (Addison and Hirsch [2], Freeman [25] and Manning [34]). As described earlier, many aspects of the strategic interaction are considered in this thesis, such as the union size, union density, wages of union and non-union workers, the firm's surplus and the union's preferences on performance-based pay.

Second, this thesis studies the strategic interaction between a union and a firm when union membership increases the bargaining power of workers, but can affect their productivity. Again, the novelty is that the strategic interaction is studied in the presence of union efficiency effects.

Third, this thesis assumes that workers are heterogeneous with respect to their skills. It studies how this heterogeneity affects the strategic interaction and union membership decisions. To the best of our knowledge, this is the first theoretic model, in the context of labor unions, that allows for union membership decisions and productivity to vary by skill. Most theoretic models analyzing labor unions assume that workers are homogenous or that heterogeneity of workers does not directly (or indirectly) affect their production (for example, heterogeneity of workers is in terms of their seniority or reputational concerns). In the real world, there is heterogeneity in skill and productivity and this has been shown to be an important factor in union membership decisions (Blanchflower [5]).

Fourth, this thesis, to the best of our knowledge, is the first to study equilibria where lower skilled workers join a union and higher skilled workers do not. This is an important contribution of the thesis because (as the following subsection explains) union membership tends to be more popular amongst workers with lower skills or productivity.

## 1.2 Motivation

A number of findings from empirical studies motivated this thesis. First, empirical studies have found associations between unions and productivity. Efficiency effects of unions vary by country, over time, and depend on the economic, institutional and political conditions under which the unions operate (Doucouliagos [21] and Tzannatos and Aidt [47]). For example, in Australia and the United Kingdom, most studies find a negative association between unions and productivity (Doucouliagos [21] and Miller and Mulvey [38]). For the United States, Doucouliagos [21] reports there was a positive

association in the late 1990s, after a negative association was found in the 1980s and early 1990s (Tzannatos and Aidt [47]).

Second, a worker's decision to become a union member depends on their individual characteristics. Workers with low skills or productivity are more likely to be members of labor unions than those with more favorable traits, holding other factors constant (Blanchflower [5]).

Third, unions that are able to influence wages typically raise wages for lower skilled workers more than for higher skilled workers (Blanchflower et al. [6], [7] and Card [19]). For example, a study by Card [19] used data from the U.S. Current Population Survey 1987–1988 to report that workers with unfavorable traits can be made better off by joining a labor union, while workers with favorable traits can be made worse off. The study of Card [19] reports a union–non-union wage gap (the difference between wages of unionized and non-union workers) of 35% for workers with the lowest quintile of observable skills and  $-10\%$  for workers with the highest quintile of observable skills. Similarly, Cai and Lui [17] report that union wage effects in Australia were significantly higher at the lower end of the wage distribution.

Fourth, the distribution of wages tends to be relatively compressed in industries with a high union density, relative to those with a low union density (Blanchflower [5], Card [18], Freeman and Medoff [26] and Kaufman [31]).

### 1.3 Outline of Thesis

Chapter 2 contains background information relevant to the thesis, including a review of the literature. Chapter 3 explores the strategic interaction between a union and workers using the Constant Wage Model. This chapter assumes that the output of workers is unobservable. Chapter 4 studies this strategic interaction when the output of workers is observable and wages depend on skill level. Chapter 5 examines this interaction when the output

of workers is unobservable for union members, but observable for non-union members. This chapter also compares the equilibrium variables across Chapters 3 through 5 and discusses the union's preferences on performance-based pay. Chapter 6 focuses on the strategic interaction between a union and a firm using the Firm Choice Model. This chapter assumes that output is unobservable. Chapter 7 examines this strategic interaction when the output of workers is observable and wages depend on skill level. Chapter 8 concludes. Appendix A contains all proofs. Appendix B extends the Constant Wage Model of Chapter 3 by endogenizing the union inefficiency effect.

# Chapter 2

## Background

This chapter contains background information relevant to the thesis. First, there is an explanation of the relevance of labor unions. Second, there is a review of the relevant literature. Third, there is an explanation of how labor unions can affect a firm's efficiency (both positively and negatively). Fourth, there is a summary of relevant empirical research on the issue.

### 2.1 The Relevance of Labor Unions

Unions remain important players in labor markets despite the level of union membership having declined in most countries over the past few decades. Union density in OECD countries averaged 28% in 2010 compared to 41% in 1990 (Booth [8]). There are a number of reasons why labor unions remain important despite this decline. These are explained in Booth [8] and are briefly summarized here.

First, for European countries, Australia and New Zealand, the influence of labor unions is better indicated by the extent of collective bargaining coverage of the workforce, rather than by union density (Booth [8]; the definition of the collective bargaining coverage rate is the number of workers covered by wage bargaining agreements as a proportion of the workforce). In France, for example, less than 8% of workers are unionized, but 90% of



its workforce is covered by union negotiated collective agreements (Booth [8]).

Second, union membership levels have remained high in a number of countries (such as Scandinavian countries) and sectors (such as the public sector). For example, in 2012 six OECD countries had union density exceeding 50%, and four OECD countries had union density exceeding two-thirds (Denmark, Finland, Iceland and Sweden) (Booth [8]).

Third, labor union influence extends beyond the direct measure of union power suggested by the union density and coverage figures (Booth [8]). For example, the threat of union organization of a non-union sector may provoke management to provide wages and working conditions that mimic those negotiated in union firms (Booth [8]).

## 2.2 Literature Review

This subsection reviews the theoretical literature on labor unions. There is a focus on those areas related to this thesis, such as union efficiency effects and the union membership decisions of workers.

### Overview of Main Theory Models

There are three main theoretic models or approaches used to analyze labor unions: the Monopoly Union Model (including its extensions, such as the Median Voter Model), the Nash bargaining approach (also known as the axiomatic or cooperative game theory approach) and the non-cooperative game theoretic approach. Most models are principally concerned with studying the interaction between the firm and the union, rather than between the union and workers. These modelling approaches are briefly outlined below to provide context. The next subsection discusses the main theories related to the union membership decisions of workers.

#### *Monopoly Union Model*

In the Monopoly Union Model, there is no bargaining between the two

parties. Instead, the union acts as a monopolist in the supply of labor and imposes a wage rate on the firm. The firm chooses the number of workers to employ at the union wage (Booth [9]).

A key extension of the Monopoly Union Model is the Median Voter Model. It builds on the Monopoly Union Model by assuming union members are heterogeneous in their preferences for a union policy, such as the wage imposed on the firm. It assumes that union leaders, who want to be re-elected, will choose the wage preferred by the median union member (Kaufman [32]). As in the Monopoly Union Model, all employed workers are also union members because the union is a monopoly provider of labor.

Some researchers argue that the Monopoly Union Model is a reasonable approximation for particular industries at particular times, however, observation seems to suggest that wages are bargained over by unions and firms (Booth [9]). The principal tool used to understand this has been game theory (Kaufman [32]).

#### *Nash Bargaining Approach*

The Nash bargaining approach (see Nash [41]), also known as the axiomatic approach or cooperative game theory approach, assumes that bargaining is a cooperative game. In other words, it assumes that the bargainers can sign a binding contract once an agreement is reached (Kaufman [32] and Manzini [35]). This approach involves the bargainers choosing the wage (or the level of employment, or both) which maximizes the Nash product (the Nash product refers to the product of the excess payoffs to the bargainers in case of agreement with respect to what they would obtain in case of disagreement) (Manzini [35]).

The Nash bargaining approach has not been widely used for modelling union and firm bargaining because it generally cannot incorporate behavioral foundations or details of the bargaining process (Kaufman [32]). There is an equivalence between the Nash bargaining outcome and strategic bargaining where two agents make alternating offers. This has been shown by Rubinstein [47].

### *Non-cooperative Game Theoretic Approach*

Non-cooperative game theory has been used for modelling of the structure of wage negotiations. In contrast to the cooperative approach, the key feature of strategic models of bargaining is the specification of the extensive form of the game, that is, the explicit modelling of the bargaining protocol (Manzini [35]). The advantage of this approach is that it explicitly models the objective functions of the bargainers, specifies the resources of the bargainers and the rules structuring the negotiations, and permits inclusion of common negotiating tactics, such as bluffing and recourse to strikes (Kaufman [32]).

The solution concept widely used in the union literature is the generalized Nash bargaining solution (Kaufman [32]). According to this, wages are determined by maximizing the product of each agent's gains from reaching a bargain, weighted by their respective bargaining strengths (Kaufman [32]).

The most common non-cooperative game theory model used in the union literature is the Right to Manage Model. It assumes that bargaining is over wages only and the firm retains authority to set the level of employment (Kaufman [32]). An extension of this model is the Efficient Contract Model which allows the bargainers to set both wages and the level of employment (Kaufman [32]).

### **The Union Membership Decision**

The key theory model of worker choices to unionize is the Social Custom Model. The Social Custom Model is primarily concerned with explaining a worker's decision to join a labor union in the presence of free-riding workers who do not join a union. In the Social Custom Model, the reason that workers join the union is to receive a good reputation (Checchi and Corneo [20]). One limitation of the Social Custom Model is that it does not consider the supply or demand for labor (Naylor [42]).

Another approach is to assume union membership is an asset that workers can buy to gain benefits that accrue from the union, such as in Pencavel

[44] (Kaufman [31]). The decision to join a union and the level of union membership can then be modelled as the outcome of a demand and supply process — workers demand union services and unions supply the services (Hirsch and Addison [29] and Kaufman [31]).

### **Union Efficiency Effects**

There is a large empirical literature on the efficiency effects of labor unions. Section 2.3 summarizes this literature.

In terms of theory, typical theoretic models of labor unions do not include union efficiency effects, even though this has been flagged as an important area for future research (see, for example, Addison and Hirsch [2], Freeman [25] and Manning [34]). According to Booth [8], this area of research has probably not progressed because of a reduction in the number of researchers focusing on the theory of labor unions over the past few decades.

### **Worker Heterogeneity**

Workers are heterogenous in productivity. In applied theoretical models, this observation may be important. Conclusions may depend critically on whether we allow for heterogeneity in the productivity of workers.

Theoretic models studying labor unions vary considerably in their assumptions on workers. Models that study bargaining between the union and firm often assume workers are homogenous. Models concerned with the union membership decision itself — such as the Median Voter Model and Social Custom Model — often assume workers are heterogeneous in some aspect.

The Median Voter Model assumes there is heterogeneity in worker preferences for a union policy — typically the wage rate bargained for. Other sources of worker heterogeneity usually relates to seniority (such as in Booth and Frank [10] and Frank [24]), value for union services (see Booth [11]), or reputation (see Booth [12]).

Levy [33] proposes a model where workers are different with respect to their suitability for a particular occupation, and studies union formation across occupations. Kaufman [31] and Kaufman [32] provide more detail on worker heterogeneity in labor union research.

## 2.3 Union Efficiency Effects

This subsection explains the number of ways a union can affect a firm’s efficiency, both positively and negatively. It is followed by a review of the empirical research on this issue. A more detailed explanation and review of the literature is available in Tzannatos and Aidt [47]. First, a union can affect a firm’s allocative efficiency if it changes the allocation of resources used to generate output. This may occur, for example, if it changes the relative union–non-union wages between sectors and types of workers (Tzannatos and Aidt [47]).

Second, a union may affect technical efficiency if it impacts a firm’s level of output given the inputs it employs. It may negatively affect technical efficiency if it encourages a restrictive practice that limits the productive use of workers, promotes inefficient hiring and firing, reduces working hours, or reduces the pace of work (Hancock et al. [28] and Tzannatos and Aidt [47]).

As reported in Hancock et al. [28], unions could improve technical efficiency if they can assist in the resolution of workplace issues that might otherwise cause productivity-diminishing behavior. For example, a union can provide workers with a means of expressing discontent as an alternative to ‘exiting’, by opening up communication channels between workers and management, and by inducing managers to alter methods of production and to adopt more efficient policies (Freeman and Medoff [26]).

Third, a union may affect dynamic efficiency if it impacts economic growth. It may do this, for example, if its activities lead to reduced investment or a slowdown in employment growth (Tzannatos and Aidt [47]).

Finally, a union could affect distributional efficiency if it results in an allocation of incomes that departs from that which would maximize social welfare (Tzannatos and Aidt [47]). This could happen, for example, if it secured higher pay for union members than for other workers simply on the basis of their being “insiders” or by changing the distribution of incomes between wages and profits (Tzannatos and Aidt [47]).

### **Empirical Findings**

A number of empirical studies have examined the efficiency effects of unions. The main results are that efficiency effects of unions vary by country and over time, and depend on the economic, institutional and political conditions under which they operate (Tzannatos and Aidt [47]).

For example, Tzannatos and Aidt [47] report that conditions that influence the union efficiency effect include: (a) the nature and enforcement of regulations governing the right of workers to organize and the way they can exercise this right, including for collective bargaining; (b) the market or welfare orientation of the government; and (c) whether the economy is competitive or not and whether it is protectionist or open to trade.

They also report that more intense product market competition tends to induce unions and management to move towards industrial relations systems which enhance the positive effects of unions and reduce the negative ones. This is confirmed by Metcalf [37] who found that negative union effects in the United Kingdom were mainly in firms that faced a small degree of product market competition.

By country, unions have been found to have an overall negative effect on productivity in the United Kingdom, Germany, Australia, New Zealand and Japan (Brunello [14] and Tzannatos and Aidt [47]). In Japan, Brunello [14] found that productivity in unionized firms was around 15% lower than in similar non-unionized firms. This finding was confirmed by Benson [4] who asked managers to rank the productivity performance of their firm relative to that of another firm in the same industry. Unions were found to reduce

the rate of return on equity by 20–25% and the ratio of profits to sales by 40% (Brunello [14]). Union presence in Japan has also been associated with a higher fraction of profits going to workers (Noda and Tachibanaki [43]). In Australia, Farmakis-Gamboni and Prentice [23] studied how union bargaining power affects the productivity of firms. They found that union bargaining power had greater negative productivity effects in sectors where unionization was more common.

Positive productivity effects have been reported for Malaysia (Standing [46]) and mixed effects were found in the United States. The evidence from the United States in the 1980s and early 1990s suggests that financial performance is better in non-unionized than unionized firms, and that the negative impact tends to be larger in industries or firms that have some monopoly power in product markets (for a survey of studies on the United States see Belman [3]). For example, Karier [30] estimated that union shares of monopoly profits may be as large as between 47% and 77%. While these figures may not be entirely representative, they do show that under certain circumstances unions are able to appropriate a substantial share of monopoly profits (Tzannatos and Aidt [47]). More recent studies of the United States suggest that the earlier negative influence of unions on profits is diminishing over time (Tzannatos and Aidt [47]).

# Chapter 3

## Constant Wage Model

### 3.1 Model Setup

Consider a sequential game involving a labor union and a finite number of workers. The set of workers,  $\mathcal{N} = \{1, 2, \dots, N_0\}$ , contains consecutive integer numbers from one to some fixed  $N_0 \geq 2$ . The model refers to a large population, so we think of  $N_0$  as a large number. Workers are indexed and ordered according to their skill; the least skillful worker is  $j = 1$  and the most skillful is  $j = N_0$ . No two workers have the same skill level. The union and the workers have common knowledge of the skill distribution, but the union cannot observe the skills of any individual worker. A worker's skill level is her private information (this may occur in, for example, advice markets (Fuchs and Garicano [27]) or in the occupations of nursing or teaching (Brekke and Nyborg [13])). Chapters 4 and 5 consider the case where the skills of individual workers are observable.

Each worker maximizes her wage, and the union maximizes the wage of union members. In terms of the timing of the game, the union first sets wages for its members subject to a budget balance condition. Following this, each worker in  $\mathcal{N}$  observes her own skill level and the union wage, and then, decides whether to join the union or not; only these two pure actions are allowed. All workers move simultaneously. Each worker considers joining



the union in the following way: given the union wage and my skill level, should I join the union or not, correctly anticipating the effect of joining on my wage and assuming that other workers play their part in equilibrium.

The objective of the union is consistent with orthodox models of labor unions, in which unions are concerned with maximizing the wages of their members (Booth [9]). While bargaining in reality may take place over a number of issues, for tractability researchers typically focus on wages (Booth [9]). Appendix B extends the model to consider bargaining over wages and working conditions. In this chapter the firm cannot take actions; a restriction also found in other theory models concerned with worker choices to unionize (for example, see Naylor [42]). Chapters 6 and 7 incorporate a firm into the model.

The workers and the firm produce an economic surplus; an assumption we make because unions can only increase wages above the competitive level if a surplus exists (Booth [9]). Implicit in this assumption is that the firm is operating in a market with limited competition. Let  $K = K(n)$  be the workers' share of the total surplus when  $n$  workers join the union. Suppose that the union negotiates with the firm regarding the share of the total surplus that all workers will obtain. This is the case in many advanced economies, where worker remuneration is often set through collective bargaining arrangements (Booth [8]). In this bargaining process, as in Nash [41],  $K(n)$  represents the relative bargaining power of workers when the union size is exactly  $n$ . Indeed, in reality the ability of labor unions to increase the surplus share that goes to workers depends to a large degree on the unionization of the workforce (Checchi and Corneo [20], Mishel [40] and Woddoups [48]).

In this thesis, the bargaining process that determines  $K(n)$  though  $n$  is exogenous. Intuitively, however, labor unions derive the legitimacy to represent the interests of the workforce at the bargaining table when they have a large number of members, either because of social norms, labor laws or the threat of more destructive industrial action (Checchi and Corneo [20]).

A benefit of using a generic and exogenous bargaining function, rather than specifying how  $K(n)$  is formed through  $n$ , is that we are able to derive qualitative results for any possible  $K(n)$  and any possible process for which  $K(n)$  is generated through  $n$ . That is, the results obtained are general in nature in the sense that they are not dependent on the properties of the bargaining process.

Assume that the bargaining function  $K : [0, N_0] \rightarrow \mathbb{R}$  is publicly observable. This is not a controversial assumption because, in reality, unions are aware of the factors influencing their bargaining position (Woddoups [48]). Suppose further that  $K : [0, N_0] \rightarrow \mathbb{R}$  is a twice differentiable function of  $n$  such that:

$$K(0) \geq 0, \quad (3.1)$$

$$K(N_0) \leq 1, \quad (3.2)$$

$$\frac{dK}{dn}(0) > \frac{1}{N_0(N_0 + 1)}, \quad (3.3)$$

$$\frac{dK}{dn}(n') > 0, \quad \text{for all } 0 \leq n' < N_0, \quad (3.4)$$

$$\frac{d^2K}{dn^2}(n') \leq 0, \quad \text{for all } 0 \leq n' \leq N_0, \quad (3.5)$$

$$\frac{dK}{dn}(N_0) = 0. \quad (3.6)$$

Assumption (3.4) states that function  $K : [0, N_0] \rightarrow \mathbb{R}$  is increasing, that is, workers' bargaining power increases as the union size increases. As described earlier, this is because the size of the union is a key determinant of its bargaining power (Mishel [40] and Woddoups [48]). Function  $K : [0, N_0] \rightarrow \mathbb{R}$  is weakly concave by assumption (3.5); that is, the marginal bargaining power is decreasing in the union size. Its maximum is achieved at  $n = N_0$ . By assumptions (3.2) and (3.4),  $K(n) \leq 1$ , for every  $n \in [0, N_0]$ . Because of assumptions (3.1), (3.2) and (3.4), then  $0 < K(n) < 1$ , for every  $n$  such that  $0 < n < N_0$ .

Assumption (3.3) helps us prove that at least some workers join the

union. Without this assumption, it would be possible that no worker would join the union in equilibrium. The intuition is that there would be no union members if the union could not sufficiently improve the bargaining power of workers, in the sense that the marginal benefit from unionization would be too small, even at the point  $n = 0$ , where the marginal bargaining power is maximized. Assumption (3.3) is remarkably weak; in particular, because we expect  $N_0$  to be large, this assumption is similar to assuming the marginal bargaining power of the first unionized worker is non-negative. Assumption (3.6) guarantees the solution to the union's problem is an interior solution. It forces the marginal bargaining power of workers to decrease to zero as the union size increases to 100% of workers. This assumption is only utilized when we want to restrict our attention to the case where not all workers are unionized.

Consider an exogenous parameter  $\varepsilon$ , named the inefficiency coefficient, such that  $0 < \varepsilon \leq 1$ . If a worker joins the union, the surplus she produces is reduced by a proportion  $\varepsilon$  with respect to how much it would otherwise be. Workers that join the union lose a fraction  $\varepsilon$  of their production because they are distracted by or committed to union activities, they have an increased ability to shirk, or they have more restrictive work rules. Because of this, a worker's decision to join the union may affect her payoff and also the payoffs of all other workers. Indeed, the trade-off faced by workers in this model is whether to join the union and increase the bargaining power of workers (that is, the "share of the pie") when unionization adversely affects their productivity (that is, the "size of the pie"). In a real situation, the inefficiency coefficient can be positive or negative (Tzannatos and Aidt [47]). Remark 1 ahead considers the case where unionized workers are more productive than non-unionized workers.

The inefficiency coefficient in this chapter is exogenous. However, the aggregate level of inefficiency in the workforce is endogenously determined because the union size is endogenously determined. While joining the union may change the bargaining power of workers and, hence, the wage level, it

does not affect the utility of union workers in other ways. That is, union workers do not gain or lose utility from, for example, attending union meetings or more restrictive work rules. Appendix B extends the model to account for this possibility. It also endogenizes the inefficiency coefficient.

The focus will be on an equilibrium where not joining the union is the choice of the most skillful; workers with skills  $j > n$  will not join the union. On the other hand, workers with skills  $j \leq n$  will join. Variable  $n$  will be determined in equilibrium. This thesis is concerned with studying the properties of such an equilibrium because, as described in Chapter 1, labor union membership tends to be more popular amongst lower skilled workers (Blanchflower [5]).

This chapter assumes that a worker's output is unverifiable or unobservable. As such, union workers receive a deterministic, constant wage; union worker of skill  $j$  earns  $\alpha$ , for a constant  $\alpha > 0$  to be determined in equilibrium. Similarly, non-union workers also earn a constant wage; a worker with skill  $j$  earns  $w \geq 0$ . In principle, this constant wage may be equal to or different from the constant wage received by union members. Because wages in aggregate cannot exceed workers' surplus (the Budget Balance constraint described below), the constant wage received by non-union workers will be equal to the average of workers' surplus after wages to union workers have been subtracted. As described in the following subsection, in equilibrium these constant wages turn out to be the same because of the incentives of workers.

Chapters 4 and 5 consider the case where output is observable or verifiable and as such the wage function depends on skills. A number of other researchers consider bargaining over constant wages, for example Booth and Frank [10]. The constant wage property may also reflect situations where output is observable, but a social norm forces the share of the surplus obtained by workers to be equal.

The workers are employed by the firm before and after their unionization decision. In other words, the level of the wage (which in our case is a

surplus) does not impact the level of employment. As with some other theory models concerned with worker choices to unionize, this model does not consider the supply and demand for labor (for example, see Naylor [42]). This assumption may be realistic for labor markets of sectors that have full employment (for example, in sectors where labor demand is sufficiently high and there are restrictions or barriers to increasing labor supply, such as through qualification, training or certification requirements).

Let  $G_U$  and  $G_N$  denote, respectively, the sum of wages to union and non-union members. Mathematically:

$$G_U = \sum_{j \in \text{union}} \alpha$$

$$G_N = \sum_{j \notin \text{union}} w.$$

Assume that the total surplus, denoted  $Y$ , is:

$$Y = (1 - \varepsilon) \sum_{j \in \text{union}} j + \sum_{j \notin \text{union}} j.$$

This specification assumes that the total surplus is additively separable along workers, so that workers' inputs are perfect substitutes. Additive separability of workers is a simplifying assumption that helps with tractability of the model.

In the proposed equilibrium, where workers with skill up to  $n$  join the union:

$$G_U(n) = \sum_{j=1}^n \alpha = \alpha n$$

$$G_N(n) = \sum_{j=n+1}^{N_0} w = (N_0 - n)w.$$

The total surplus becomes:

$$Y(n) = (1 - \varepsilon) \sum_{j=1}^n j + \sum_{j=n+1}^{N_0} j = \sum_{j=1}^{N_0} j - \varepsilon \sum_{j=1}^n j.$$

Hence:

$$Y(n) = \frac{N_0(N_0 + 1) - \varepsilon n(n + 1)}{2}. \quad (3.7)$$

The firm surplus and workers' surplus are denoted  $Y_f$  and  $Y_w$ , respectively, and are defined as  $Y_f(n) = [1 - K(n)]Y(n)$  and  $Y_w(n) = K(n)Y(n)$ . Hence, the workers' surplus is:

$$Y_w(n) = \frac{K(n)}{2} [N_0(N_0 + 1) - \varepsilon n(n + 1)].$$

The total expenditure on wages for all workers must be equal to the workers' surplus. This "Budget Balance" constraint is:

$$Y_w(n) = G_U(n) + G_N(n). \quad (3.8)$$

Therefore:

$$G_U(n) + G_N(n) = Y_w(n) = K(n)Y(n) = \frac{K(n)}{2} [N_0(N_0 + 1) - \varepsilon n(n + 1)].$$

Because all workers in the proposed equilibrium earn the same wage, the amount of the total surplus received by union workers is given by  $G_U(n) = n\alpha$ ; the amount received by non-unionized workers is  $G_N(n) = (N_0 - n)\alpha$ ; and the workers' surplus becomes:

$$Y_w(n) = G_U(n) + G_N(n) = \alpha N_0.$$

Hence, the sum of all wages paid to workers,  $G_U(n) + G_N(n)$ , becomes:

$$\alpha N_0 = G_U(n) + G_N(n) = \frac{K(n)}{2} [N_0(N_0 + 1) - \varepsilon n(n + 1)]. \quad (3.9)$$

The wage  $\alpha$  is:

$$\alpha = \frac{K(n)[N_0(N_0 + 1) - \varepsilon n(n + 1)]}{2N_0}. \quad (3.10)$$

Everything else constant, the wage  $\alpha$  increases with the bargaining power of workers,  $K$ , and decreases with the inefficiency of union members,  $\varepsilon$ . As the number of union members  $n$  increases,  $K(n)$  also increases, but the other factor in the numerator,  $N_0(N_0 + 1) - \varepsilon n(n + 1)$ , decreases.

Consider the function  $\alpha = \alpha(n)$  defined by equation (3.10). This function is maximized at  $n^*$ , with  $\alpha^* = \alpha(n^*)$ . Value  $\alpha^*$  is going to be the equilibrium wage of workers and  $n^*$  the equilibrium union size. For every  $\alpha < \alpha^*$ , the inverse image of  $\alpha$ , for any  $\alpha < \alpha^*$ , has at most two elements. This is true because  $\alpha(n)$  is continuous and strictly concave (by Lemma 1 ahead). Define  $n_1(\alpha)$  as: (i) the smaller of these two elements, if there exists  $n < n^*$  such that  $\alpha = \alpha(n)$ ; or (ii)  $n_1(\alpha) = 0$ , if there does not exist  $n < n^*$  such that  $\alpha = \alpha(n)$ .

A strategy profile specifies the strategy of each player. First, it specifies the wage,  $\alpha \in [0, \infty)$ , which is chosen by the union. The union chooses the salary of its members,  $\alpha \geq 0$ , subject to constraint (3.8), anticipating how each possible wage choice will change the workers' decisions. Second, the strategy profile specifies for each worker  $j$ , their action,  $a_j$ , which specifies whether they join the union or not, for each announced wage for union members, as follows.

$$a_j = \begin{cases} \text{unionize, if } [\alpha < \alpha^* \text{ and } j \leq n_1(\alpha)], \text{ or } [\alpha = \alpha^* \text{ and } j \leq n^*], \\ \text{or } [\alpha > \alpha^*] \\ \text{not unionize, if } [\alpha < \alpha^* \text{ and } j > n_1(\alpha)], \text{ or } [\alpha = \alpha^* \text{ and } j > n^*] \end{cases}$$

A strategy profile is an equilibrium if: (a) no worker has a unilateral profitable deviation, assuming all other players are playing their part in this equilibrium; and (b) the union's choice maximizes the wage of union

members, under constraint (3.8), when all workers play their prescribed actions in equilibrium.

In the case where the union cannot honor its wage bill, the union will try to fulfil its promised wage until it has exhausted the workers' surplus. This will only occur if a worker plays a non-equilibrium action or the union offers a wage that is too high (that is, if  $\alpha > \alpha^*$ ). In this case, all union workers receive a wage that is smaller than promised until the workers' surplus is exhausted, even if it leaves non-union workers with no share of the surplus. In other words, the union can use all of the workers' surplus to pay its members, such that the Budget Balance constraint holds.

As described earlier, this thesis is concerned with studying the properties of an equilibrium where the lower skilled workers join the union. There may exist other equilibria where workers are not the lowest skilled. Identifying all the possible equilibrium combinations of wages and union member skill distributions is generally outside the scope of this thesis, but, as an example, an alternative equilibrium is discussed in Remark 2 ahead.

## 3.2 Equilibrium

In the proposed equilibrium, all workers earn the same wage regardless of their unionization status. That is, the constant wage received by non-union workers,  $w$ , is the same as the constant wage of union workers,  $\alpha$ , even though they are relatively more productive. Because non-unionized workers can obtain wage  $\alpha$  if they join the union, the firm is paying these workers their outside option and not a cent more. The firm is unable to distinguish workers by their output; so more productive workers cannot enforce a wage schedule that depends on their output or skill. The firm chooses to pay non-unionized workers the wage  $\alpha$  because, if it did not, they would join the union and the total surplus and the profit of the firm would be reduced. This is the so-called *threat effect* (Farber [22]).

The problem of the union is to maximize the wage,  $\alpha$ , of its members



subject to condition (3.10). As described in Chapter 2, the assumption that the union is interested in maximizing the wages of its members is generally not controversial. The union anticipates the workers' union membership decisions in the spirit of backward induction. In particular, it knows for each wage the corresponding number of workers that will join. Mathematically, the union's problem is:

$$\max_{\alpha \geq 0} \alpha \quad \text{subject to condition (3.10).}$$

The next result explains how the wage  $\alpha$  depends on the number of unionized workers.

**Lemma 1** *The function  $\alpha = \alpha(n)$  in equation (3.10) is strictly concave.*

The wage  $\alpha$  is a function of the number of union members  $n$  (by equation 3.10); and this function is strictly concave (by Lemma 1). In order to maximize  $\alpha$ , the union chooses a suitable number of members,  $n$ . Depending on parameters, the union chooses the upper corner solution,  $n = N_0$ , or the unique  $n^* \in (0, N_0)$  that satisfies the first order condition. This unique interior solution,  $n^*$ , is implicitly defined by:

$$\frac{d\alpha(n^*)}{dn} = 0 \quad \Leftrightarrow \quad \frac{dK(n^*)}{dn} \frac{n^*}{K(n^*)} = \frac{\varepsilon n^*(2n^* + 1)}{N_0(N_0 + 1) - \varepsilon n^*(n^* + 1)}. \quad (3.11)$$

In particular,  $n^* > 0$ ; that is, there are always at least some workers in the union, regardless of the inefficiency coefficient of the union. This is a consequence of assumption (3.3), which implies that the marginal bargaining power when  $n = 0$  is sufficiently large to provide incentives for at least one worker to enter the union. This assumption drives the lowest skilled worker into the union because it ensures the impact on her wage from improved bargaining power more than offsets the negative efficiency impact of joining the union.

The left-hand side of condition (3.11) is the elasticity of the bargaining function with respect to the union size, calculated at  $n^*$ . Let  $\gamma(n)$  denote

the elasticity of the bargaining function with respect to union size  $n$ , mathematically:

$$\gamma(n) = \frac{dK(n)}{dn} \frac{n}{K(n)}.$$

The equation on the right of equivalence (3.11) characterizes the union size  $n^*$  that maximizes the workers' surplus. Indeed, as  $Y_w = \alpha N_0$  and  $N_0 > 0$  is a positive constant, then the value of  $n$  that maximizes  $\alpha$  must be the same as the one maximizing  $Y_w$ .

At the optimal union size,  $n^*$ , the elasticities of  $K$  and  $Y$  with respect to  $n$  have the same absolute value. As  $Y_w = KY$ , then  $dY_w/dn = KdY/dn + YdK/dn$  and:

$$\begin{aligned} \frac{dY_w}{dn} = 0 & \Leftrightarrow K(n) \frac{dY}{dn} + Y(n) \frac{dK}{dn} = 0, \\ \frac{d\alpha}{dn} = 0 & \Leftrightarrow \frac{dY_w}{dn} = 0 \Leftrightarrow \frac{dK}{dn} \frac{n}{K(n)} = -\frac{dY}{dn} \frac{n}{Y(n)}. \end{aligned}$$

The right-hand side of the equation on the right of equivalence (3.11) is equal to the absolute value of the elasticity of the total surplus,  $(dY/dn)/(Y/n)$ , while the left-hand side represents the elasticity of the bargaining power of workers with respect to the union size,  $(dK/dn)/(K/n)$ . The union size  $n = n^*$  maximizes the workers' surplus,  $Y_w$ . In other words, the equation on the right-hand side of equivalence (3.11) characterizes the union size that maximizes the workers' surplus.

As union membership increases, the bargaining power of workers,  $K(n)$ , increases, but the total output,  $Y(n)$ , decreases. If the union size,  $n$ , is sufficiently small, an increase in the union membership increases the workers' surplus because the positive effect on  $K(n)$  is larger than the absolute value of the negative effect on  $Y(n)$ . At  $n = n^*$  these two opposing effects have the exact same strength. If  $n$  continues to increase beyond  $n^*$ , then the negative effect dominates the positive one, and the workers' surplus and wages start to decrease.

The proposed strategy profile is an equilibrium. If the union chooses  $\alpha > \alpha^*$ , all workers will join the union and share equally the workers' surplus. In this case, the union will consume the whole workers' surplus. There is no profitable deviation for a union worker because, if she leaves the union, the union will redistribute the workers' surplus amongst the remaining union members and she will obtain zero outside of the union.

The union has no incentive to offer such a wage,  $\alpha > \alpha^*$ . This is because such a wage cannot be honored and the realized wage of union members,  $\omega(N_0) = Y_w(N_0)/N_0$ , will be smaller than  $\alpha^*$  (because the function  $Y_w(n)$  is decreasing for all  $n > n^*$ ):

$$\omega(N_0) = \frac{Y_w(N_0)}{N_0} < \frac{Y_w(n^*)}{N_0} = \alpha^*.$$

Similarly, if the union chooses  $\alpha < \alpha^*$ , there will be  $n_1(\alpha)$  union members. In this case, no union member has an incentive to leave because it will decrease their wage; the workers are already short of the wage-maximizing union membership size  $n^*$  (the wage is increasing in the union size for  $n < n^*$ ). Likewise, non-union workers have no incentive to join because they will earn the same wage inside and outside of the union. The union clearly does not have incentives to offer this type of wage because the wage of union members will be smaller than  $\alpha^*$ .

When the union chooses  $\alpha^*$ , workers not in the union cannot find a profitable deviation when all other players play their part in the equilibrium; it does not matter if they stay out of the union or not; they always obtain  $\alpha^*$ . The only non-trivial incentive condition is related to the incentives of union workers. If a union member deviates and leaves the union, the bargaining power of workers falls from  $K(n)$  to  $K(n-1)$ , but the total surplus,  $Y$ , rises because the deviating worker's production increases by a fraction  $\varepsilon$ . The next proposition proves that this non-trivial incentive condition is satisfied. Union members do not have incentives to deviate, so the proposed strategy profile is indeed an equilibrium. This result also characterizes this equilibrium. All proofs are in Appendix A.

**Proposition 1** *Suppose that assumptions (3.1) through (3.5) are satisfied. Then, there exists an equilibrium where the  $n^*$  workers of lowest skill,  $0 \leq n^* \leq N_0$ , join the union, and the  $N_0 - n^*$  workers of highest skill do not join the union. The number of union members in this equilibrium,  $n^*$ , is uniquely determined by the unique solution of the union's problem. The equilibrium union size is either the upper corner  $n = N_0$  (all workers are in the union), or the unique interior solution implicitly defined by the first order condition of the union's problem, the equivalent equations (3.11).*

*Moreover, all workers are unionized ( $n^* = N_0$ ) if and only if the inefficiency coefficient is sufficiently small in the precise sense that:*

$$\varepsilon \leq \frac{(N_0 + 1)\gamma(N_0)}{(N_0 + 1)\gamma(N_0) + 2N_0 + 1}, \quad (3.12)$$

*where  $\gamma(N_0)$  represents the elasticity of the workers' bargaining power,  $K$ , with respect to the number of union members,  $n$ , calculated at  $n = N_0$ . Formally:*

$$\gamma(N_0) = \frac{dK(N_0)}{dn} \frac{N_0}{K(N_0)}.$$

**Remark 1** *Suppose the unionized workers are equally or more productive than non-unionized workers, that is, if  $\varepsilon \leq 0$ . In this case, all workers will be in the union because it advances all workers. From now on we disregard this case because it is trivial.*

The equilibrium outlined in Proposition 1 shares characteristics with unionized industries in reality. In particular, there is a high degree of wage compression because all workers receive the same wage. Also, lower skilled workers are unionized and higher skilled workers are not, which matches the mix of skills observed across union and non-union labor described in Chapter 1 (see Blanchflower [5]).

The right-hand side of equation (3.11) is increasing in the inefficiency coefficient,  $\varepsilon$ , and decreasing in the workforce size,  $N_0$ . However, as these variables change, the equilibrium number of unionized workers,  $n^*$ , will also

change. When not all workers are unionized, inequality (3.11) states that the optimal number of union members depends on the elasticity of the workers' bargaining power, the inefficiency coefficient of the union, and the size of the workforce.

Proposition 1 also computes a full unionization condition. The right-hand side of inequality (3.12) is an increasing function of  $\gamma(N_0)$ , which is the elasticity of the workers' bargaining power with respect to the number of unionized workers, calculated at  $N_0$ . The larger the elasticity, the more likely full union membership becomes, for any fixed inefficiency,  $\varepsilon$ . The right-hand side of inequality (3.12) grows from zero, when  $\gamma(N_0) = 0$ , to 100% when  $\gamma(N_0) \rightarrow +\infty$ . Intuitively, if the benefit from joining the union is large relative to the cost, then more workers will join the union. If the bargaining elasticity (when  $N_0$  workers join the union) is sufficiently large relative to the inefficiency coefficient, as given in Proposition 1, then all workers decide to unionize.

The right-hand side of inequality (3.12) is a decreasing function of workforce size,  $N_0$ . This implies that, for any fixed inefficiency,  $\varepsilon$ , as the population of workers increases and  $\gamma(N_0)$  remains fixed, it becomes less likely to find full union membership. In the limit, as the population of workers grows large,  $N_0 \rightarrow +\infty$ , the full union membership condition (3.12) converges to:

$$\varepsilon \leq \frac{\gamma_\infty}{\gamma_\infty + 2}, \quad \text{where } \gamma_\infty = \lim_{N_0 \rightarrow +\infty} \gamma(N_0) \quad (3.13)$$

This inequality illustrates an important result: it is possible for a workforce to be fully unionized even at high levels of inefficiency, so long as the union can raise the workers' bargaining power sufficiently.

Next, we return to the case of a finite, fixed number of workers,  $N_0$ , and consider the case where assumption (3.6) holds. This assumption requires the marginal bargaining power of workers to decrease to zero as union membership grows to 100%. Mathematically, it requires  $\gamma(N_0) = 0$ , or equivalently,  $dK(N_0)/dn = 0$ . In this case, the unique solution  $n^*$  is interior, or put another way, assumption (3.6) rules out the possibility of an upper corner

solution for the union's problem, as the next result explains. The intuition is that the most productive worker has nothing to gain from joining the union. She also has the most to lose from joining the union because she is the most productive of all workers.

**Corollary 1** *Suppose that assumptions (3.1) through (3.5) hold. In addition, suppose that assumption (3.6) holds. Then, some workers are not union members, regardless of the exact value of the inefficiency coefficient of the union,  $\varepsilon$ . The number of union members is the implicit solution  $n^*$  of equation (3.11), with  $0 < n^* < N_0$ .*

The equilibrium discussed so far has been robust to unilateral deviations. However, because unions are essentially a collection or coalition of workers, the concept of a unilateral deviation may not be the most relevant. Suppose that workers can form coalitions,  $\zeta$ , in order to deviate collectively from the proposed strategy profile to gain a higher salary. The next proposition proves that the equilibrium outlined in Proposition 1 is robust to coalition deviations; that is, no coalition of workers has a profitable deviation from this equilibrium.

**Proposition 2** *Suppose that assumptions (3.1) through (3.5) hold and consider the equilibrium outlined in Proposition 1. In this equilibrium, no coalition of workers has a profitable deviation from the proposed strategy profile.*

There are three possible types of coalitions: (a) a coalition of entirely non-union workers, (b) a coalition of entirely union workers, and (c) a coalition containing both union and non-union workers. In equilibrium, no coalition that consists entirely of non-union workers has a profitable deviation because each of these workers obtains  $\alpha^*$  outside the union and  $\alpha^*$  inside the union.

Similarly, no coalition that consists entirely of union workers has a profitable deviation. If union workers leave the union, the impact on wages from lower bargaining power more than offsets the benefit of increased efficiency.

No coalition containing both union and non-union workers has a profitable deviation. If a non-union worker and union worker switch their membership status, the bargaining power remains unchanged, but output falls by  $\varepsilon$  times the absolute value of the difference in skills between these two workers.

We also checked the robustness of the equilibrium to a different mix of skills across union and non-union labor. The next proposition shows that no equilibrium exists in which higher skilled workers (those with skills between  $n$  and  $N_0$ ) are unionized, while lower skilled workers (those with skills no greater than  $n$ ) are not in the union.

**Proposition 3** *Suppose assumptions (3.1) through (3.5) are satisfied. Suppose further that high skilled workers (those with skills between  $n$  and  $N_0$ ) are in the union, while lower skilled workers (those with skills no greater than  $n$ ) are not in the union. This proposed strategy profile is not an equilibrium; there exists profitable unilateral and coalitional deviations.*

The result of Proposition 3 suggests that, with the current model setup, lower skilled workers prefer to be union members more than higher skilled workers; a finding which is consistent with empirical studies discussed in Chapter 1. The main factor driving the result is that there are profitable deviations for both union and non-union workers. If a union worker and non-union worker switch their membership status it leads to an increase in the total surplus and the workers' surplus. This is because the aggregate efficiency increases (since output is proportional to skill and union workers are higher skilled than non-union workers), while the bargaining power is unchanged.

The next corollary is concerned with the comparative statics of the equilibrium. In particular, it studies how the equilibrium number of union members, salary, workers' surplus, firm surplus and total surplus respond to changes in the inefficiency coefficient. Because the corollary expresses some conditions in terms of elasticities we define these here. Recall that  $\gamma(n)$  is

the elasticity of the bargaining function with respect to  $n$ , and let  $E_{K,n}$  be the elasticity of  $dK/dn$  with respect to  $n$ . Clearly, the elasticity  $\gamma(n)$  is positive and represents the percentage increase in the bargaining function when the union size increases by 1%. The elasticity  $E_{K,n}$  is negative because of assumption (3.5) of diminishing marginal returns. Indeed, this elasticity represents the percentage decrease in  $dK/dn$  when the union size increases by 1%.

Recall from the definition of  $K(n)$  that we think of  $K$  as a function of the union size,  $n$ , alone. When the inefficiency coefficient changes, the bargaining power of workers may change because  $n^*$  may change. That is, the inefficiency coefficient affects  $K$  indirectly, not directly. The proof of the next result relies on this fact.

**Corollary 2** *Suppose assumptions (3.1) through (3.6) are satisfied. In this case, the equilibrium constant wage,  $\alpha^*$ , and the equilibrium number of union members,  $n^*$ , are decreasing functions of the inefficiency coefficient,  $\varepsilon$ ; that is, as the union inefficiency rises, the wage and the number of union members fall. Mathematically:*

$$\frac{d\alpha^*}{d\varepsilon} < 0, \quad \frac{dn^*}{d\varepsilon} < 0.$$

*The equilibrium workers' surplus,  $Y_w^*$ , is always decreasing in the inefficiency coefficient, evident by  $d\alpha^*/d\varepsilon < 0$ . The extent to which the equilibrium total surplus,  $Y^*$ , and firm surplus,  $Y_f^*$ , respond to changes in the inefficiency coefficient depends on the bargaining function,  $K(n)$ . Mathematically:*

$$\begin{aligned} \frac{dY_w^*}{d\varepsilon} &< 0, \\ \frac{dY^*}{d\varepsilon} &< 0 \quad \Leftrightarrow \quad \gamma(n^*) > 1 + E_{K,n} \frac{dK(n^*)}{dn} \frac{N_0(N_0+1) - \varepsilon n^*(n^*+1)}{\varepsilon(2n^*+1)K(n^*)}, \\ \lim_{N_0 \rightarrow +\infty} \frac{dY_f^*}{d\varepsilon} &< 0 \quad \Leftrightarrow \quad -\gamma(n^*) > E_{K,n} + \frac{2K(n^*)}{1-K(n^*)} - \frac{2\varepsilon n^{*2}}{N_0^2 - \varepsilon n^{*2}}. \end{aligned}$$



The first result of the corollary (that the equilibrium wage and union size are decreasing in the inefficiency coefficient) may initially seem obvious. However, this is not quite the case — workers in the model face a trade-off as the inefficiency rises. In particular, as the inefficiency coefficient rises, workers can increase their aggregate efficiency by leaving the union (a positive effect on wages), but doing so will reduce their bargaining power (a negative effect on wages). In other words, both the size of the surplus and share of the surplus are affected in opposite directions as the inefficiency rises. This trade-off makes changes in wages and the level of union membership ambiguous. The net effect is negative for both wages and union membership.

The proof of  $dn^*/d\varepsilon < 0$  in Appendix A showed that this result holds for any specification of the surplus,  $Y$ , so long as it had the following properties:  $Y_\varepsilon < 0$ ,  $Y_n < 0$ ,  $Y_{nn} \leq 0$  and  $Y_{n\varepsilon} \leq 0$ . The terms  $Y_\varepsilon$  and  $Y_n$  are the partial derivatives of the surplus with respect to the inefficiency coefficient and union size, respectively. The term  $Y_{nn}$  is the second order partial derivative of the surplus with respect to the union size. The term  $Y_{n\varepsilon}$  is the cross partial derivative of the surplus with respect to the union size and inefficiency coefficient. That is, the result that the union size is decreasing in the inefficient coefficient is quite general in the sense that it holds for any specification of the surplus, so long as it decreases in the union size and inefficiency coefficient and does so at a decreasing rate (i.e. there is a diminishing marginal effect on the surplus from unionization).

The equilibrium workers' surplus is negatively affected by a higher inefficiency, as evident by the result that  $d\alpha^*/d\varepsilon < 0$  and because  $Y_w = \alpha N_0$ . When the inefficiency rises, it leads to a fall in the workers' bargaining power (that is, the "share of the pie"). Depending on the bargaining function, this negative effect either more than offsets a rise in total surplus or is complemented by a fall in total surplus. In other words, the magnitude of the fall in workers' surplus may also reflect changes in the total surplus (the "size of the pie").

The equilibrium total surplus may increase or decrease with the ineffi-

ciency depending on the bargaining function. This is because there are positive and negative effects from a higher inefficiency. In particular, a higher inefficiency makes union members less efficient (a negative direct effect on the surplus), but at the same time decreases the size of the union (a positive indirect effect on the surplus). The overall effect depends on the rate at which workers leave the union when the inefficiency rises, and this depends on how leaving the union affects their bargaining.

The result shows that if the elasticity  $\gamma(n^*)$  is sufficiently small, then a higher inefficiency will increase the equilibrium total surplus. The intuition is that (when  $\gamma(n^*)$  is sufficiently small), many workers will leave the union when the inefficiency rises because it does not adversely affect their bargaining power too much (because the elasticity of the bargaining function is too small). The sharp fall in union membership increases the total surplus.

Similarly, the equilibrium firm surplus may increase or decrease with the inefficiency depending on the bargaining function. The higher inefficiency leads to an increase in the firm's bargaining power, which has a positive effect on its surplus. This may be offset by a fall in the total surplus, or complemented by a rise in the surplus depending on the bargaining function,  $K$ . Indeed, this result suggests there exist situations where the firm has incentives to increase the inefficiency of union workers. By doing so, workers leave the union, reducing its bargaining power and increasing the size of the surplus accrued by the firm. Moreover, the firm only prefers a lower inefficiency so long as it does not lead to a much larger union size. In contrast, workers always prefer lower inefficiency, so they can produce more surplus and join the union to improve their bargaining power.

Because  $dY/d\varepsilon = dY_f/d\varepsilon + dY_w/d\varepsilon$ , and because Corollary 2 established that  $dY_w^*/d\varepsilon < 0$ , it is clear that:

- if the equilibrium total surplus is increasing in the inefficiency coefficient, then the equilibrium firm surplus must also be increasing in the inefficiency coefficient (that is, if  $dY^*/d\varepsilon > 0$ , then  $dY_f^*/d\varepsilon > 0$ ).
- if the equilibrium firm surplus is decreasing in the inefficiency coefficient,

cient, then the equilibrium total surplus must also be decreasing in the inefficiency coefficient (that is, if  $dY_f^*/d\varepsilon < 0$ , then  $dY^*/d\varepsilon < 0$ ).

The model so far has assumed that the union bargains with the firm over the entire surplus. However, the firm may be inefficient in production (for example, if it is a monopoly) and require part of the surplus to break even (or, to put another way, the firm may face a fixed cost). Let  $M$  be the amount of the surplus that the firm requires to break even. In this case, suppose that the union bargains with the firm over the excess surplus  $Y - M$ , rather than the entire surplus  $Y$ . The workers' surplus would become  $Y_w = K(n)(Y - M)$ . The model studied so far has simply been the case where  $M = 0$ . If  $M > 0$ , the equilibrium equation (3.11) that defines the union size changes. Specifically, the denominator of the right-hand side of the equilibrium equation (3.11) needs to be subtracted by  $M$ . Mathematically, equation (3.11) becomes:

$$\frac{dK(n^*)}{dn} \frac{n^*}{K(n^*)} = \frac{\varepsilon n^*(2n^* + 1)}{N_0(N_0 + 1) - \varepsilon n^*(n^* + 1) - M}.$$

By making this subtraction, the denominator of the right-hand side decreases (by  $M$ ), and then, the right-hand side increases. The equation on the left-hand side of this equality (the elasticity of the bargaining function) does not depend on  $M$ . Consequently, when  $M$  increases, the equilibrium union size  $n^*$  decreases. This is expected — if there is less for the workers to gain from bargaining and the cost of bargaining remains unchanged, then the optimal decision of the union is to choose a smaller union size  $n^*$ . In other words, the equilibrium surplus, union size and union density are decreasing in the amount of the surplus that the firm requires to break even.

As described earlier, there may exist other equilibria where workers are not the lowest skilled. Identifying all the possible equilibrium strategy profiles is generally outside the scope of this thesis, but, as an example, an alternative equilibrium is discussed in the following remark.

**Remark 2** Consider the following strategy profile, which is almost identical to the one described earlier; the only difference is that worker  $n^* - 1$  does not join the union when the union offers the equilibrium wage  $\hat{\alpha}$ . The strategy profile defines the wage,  $\alpha \in [0, \infty)$ , chosen by the union. It also defines, for each worker  $j$ , their action,  $a_j$ , which specifies whether they join the union or not for each announced wage for union members, as follows.

$$a_j = \begin{cases} \text{unionize, if } [\alpha < \hat{\alpha} \text{ and } j \in \{1, 2, \dots, n_1(\alpha) - 2, n_1(\alpha)\}], \\ \text{or } [\alpha = \hat{\alpha} \text{ and } j \in \{1, 2, \dots, n^* - 2, n^*\}], \text{ or } [\alpha > \hat{\alpha}] \\ \\ \text{not unionize, if } [\alpha < \hat{\alpha} \text{ and } j \in \{n_1(\alpha) - 1, n_1(\alpha) + 1, \dots, N_0\}], \\ \text{or } [\alpha = \hat{\alpha} \text{ and } j \in \{n^* - 1, n^* + 1, \dots, N_0\}] \end{cases}$$

This strategy profile is an equilibrium. If the union chooses  $\hat{\alpha}$ , no union worker has an incentive to deviate because the union is already one member short of the wage-maximizing union membership level (as described earlier, the wage is increasing in union membership for  $n < n^*$ ). Non-unionized workers do not have a profitable deviation because they earn the same wage as inside the union.

If the union chooses  $\alpha > \hat{\alpha}$ , all workers will join the union and share equally the workers' surplus. In this case, there is no profitable deviation for a union worker; the union will consume the whole workers' surplus. The union has no incentive to offer such a wage,  $\alpha > \hat{\alpha}$ , because it cannot be honored and the realized wage of union members will be smaller than  $\hat{\alpha}$  (as described in the text above Proposition 1).

Similarly, if the union chooses  $\alpha < \hat{\alpha}$ , there will be  $n_1(\alpha) - 1$  union members. In this case, no union member has an incentive to deviate because it will decrease their wage; the union is already short of the wage-maximizing union membership size (the wage is increasing in the union size for  $n < n^*$ ). Likewise, non-union workers have no incentive to join because they earn the same wage inside and outside of the union. The union clearly does not have incentives to offer this type of wage because the wage of union members will

be smaller than  $\hat{\alpha}$ .

Hence,  $\{a_j\}$  for  $j \in \mathcal{N}$ , and  $\hat{\alpha}$  form a subgame perfect equilibrium, where workers with skills  $j = n^* - 2$  and  $j = n^*$  belong to the union, but worker with skill  $j = n^* - 1$  does not.

The model so far has considered a general bargaining function,  $K(n)$ . As a result, some of the equilibrium variables have been difficult to solve as functions of exogenous parameters only (for example, the equilibrium union size,  $n^*$ ). For these variables, we instead defined conditions which they must satisfy in equilibrium (for example, condition 3.11 which defines  $n^*$ ). The next two subsections overcome this limitation by restricting the bargaining function. This allows us to solve explicitly for the equilibrium variables as functions of exogenous parameters only. It also allows us to gain insight compared to the case where the bargaining function is generic.

### 3.3 Bargaining with Constant Elasticity

This subsection studies the Constant Wage Model under an assumption that the elasticity of the bargaining function, with respect to union size, is constant. The restriction may be realistic if there are labor laws, social norms or political reasons that force the bargaining power of workers to change with the union density in a constant way. Fix constants  $0 < \gamma < 1$  and  $0 < \kappa \leq 1$ . Suppose that the bargaining function is  $K(n) = \kappa n^\gamma / N_0^\gamma$ , for all  $n \in [0, N_0]$ . As before, the bargaining function is increasing and concave. For every  $n \in [0, N_0]$ , the elasticity of the bargaining function with respect to the union size is always equal to:

$$\gamma = \frac{dK(n)}{dn} \frac{n}{K(n)}.$$

Because  $K(n)$  is an increasing function, the parameter  $\kappa$  is the maximum share of the surplus that the union can obtain. Name this parameter as the

maximum union share. The next proposition defines the equilibrium size of the union.

**Proposition 4** *Suppose that  $K(n) = \kappa n^\gamma / N_0^\gamma$ , for all  $n \in [0, N_0]$ . In this case, the optimal number of workers for the union is:*

$$n^* = \frac{\gamma + 1}{2\gamma + 4} \left( -1 + \sqrt{1 + \frac{4N_0(N_0 + 1)\gamma(\gamma + 2)}{\varepsilon(\gamma + 1)^2}} \right). \quad (3.14)$$

*In particular,  $n^*$  does not depend on the maximum union share,  $\kappa$ .*

A straightforward computation reveals that  $dn^*/d\gamma > 0$ . The number of union members,  $n^*$ , is an increasing function of the elasticity of the workers' bargaining power with respect to the number of union members,  $\gamma$ . This is because, when the bargaining function is more elastic, raising the union size leads to greater gains in bargaining power (compared to when it is less elastic). As that elasticity decreases to zero,  $\gamma \rightarrow 0$ , then the equilibrium union size decreases to zero,  $n^* \rightarrow 0$ . At the other extreme, as the elasticity grows and  $\gamma \rightarrow 1$ , then:

$$\lim_{\gamma \rightarrow 1} n^* = \frac{1}{3} \left( -1 + \sqrt{1 + \frac{3N_0(N_0 + 1)}{\varepsilon}} \right).$$

In particular, as the elasticity approaches one,  $\gamma \rightarrow 1$ , and the workforce size grows large,  $N_0 \rightarrow +\infty$ , the equilibrium union density,  $n^*/N_0$ , converges to  $1/\sqrt{3\varepsilon}$ . This result illustrates that even with a large inefficiency coefficient the union density can still be quite large so long as the elasticity is large. In the most extreme case, union workers may produce no surplus,  $\varepsilon = 1$ , and the union density is 58%.

**Corollary 3** *Suppose that  $K(n) = \kappa n^\gamma / N_0^\gamma$ , for all  $n \in [0, N_0]$ . All workers are unionized ( $n^* = N_0$ ) if and only if the inefficiency is sufficiently small in the sense that:*

$$\varepsilon \leq \frac{\gamma(N_0 + 1)}{(\gamma + 1) \left[ N_0 \left( \frac{\gamma + 2}{\gamma + 1} \right) + 1 \right]}.$$

A straightforward computation reveals that this level of inefficiency is the same as in equation (3.12) when  $\gamma = \gamma(N_0)$ . In other words, this result matches the result in the previous section where the bargaining function was generic, as expected. The next result is concerned with the comparative statics.

**Corollary 4** *Suppose that the bargaining function has constant elasticity  $\gamma$  with regard to the union size. In equilibrium, the workers' surplus,  $Y_w^*$ , is always decreasing in the inefficiency coefficient, the total surplus,  $Y^*$ , does not change with the inefficiency coefficient, and the firm surplus,  $Y_f^*$ , increases with the inefficiency coefficient. Mathematically:*

$$\frac{dY_w^*}{d\varepsilon} < 0, \quad \lim_{N_0 \rightarrow +\infty} \frac{dY^*}{d\varepsilon} = 0, \quad \lim_{N_0 \rightarrow +\infty} \frac{dY_f^*}{d\varepsilon} > 0.$$

The equilibrium total surplus is unchanged if the inefficiency coefficient rises; with a fall in the workers' surplus exactly offsetting a rise in the firm's surplus. The equilibrium workers' surplus declines as the inefficiency rises because the higher inefficiency reduces the union size and hence the workers' bargaining power. The combination of an unchanged total surplus and the lower bargaining power of workers leads to a rise in the firm's surplus. This result suggests that the constant elasticity bargaining function allows us to study how the inefficiency affects the surplus allocation, rather than the size of the surplus. In particular, this result suggests that the firm has incentives to increase the inefficiency of union workers. By doing so, workers leave the union, reducing its bargaining power and increasing the size of the surplus accrued by the firm. In contrast, workers prefer lower inefficiency, so they can produce more surplus and join the union to improve their bargaining power.

As the elasticity of the bargaining power,  $\gamma$ , increases, the equilibrium bargaining power of workers rises indirectly because the number of union members,  $n^*$ , rises;  $dn^*/d\gamma > 0$ . However, the higher elasticity has a direct negative effect on the workers' bargaining power. If unions can change

their bargaining function, they need to consider both the direct and indirect effects. The direct effect refers to the partial derivative  $\partial K/\partial\gamma$ , which is negative. The net effect can be positive or negative depending on the exogenous parameters. For example, if  $\varepsilon = 1/2$  and  $N_0 = 1000$ , then the sign of the derivative is positive when  $\gamma = 2/3$  and negative when  $\gamma = 1/2$ . The indirect effect refers to the product  $(dK/dn)(dn^*/d\gamma)$ , which is always positive. Mathematically, the change is given by the following derivative:

$$\frac{dK(n^*)}{d\gamma} = \frac{\partial K}{\partial\gamma} + \frac{dK}{dn} \frac{dn^*}{d\gamma}.$$

**Example 1** *Consider the constant elasticity bargaining function above and let  $\gamma = 1/2$ . In this case, solving equation (3.11) for the equilibrium union size  $n^*$  results in:*

$$n^* = \frac{3}{10} \left( -1 + \sqrt{1 + \frac{20}{9} \left( \frac{N_0(N_0 + 1)}{\varepsilon} \right)} \right).$$

*Moreover, all workers are unionized ( $n^* = N_0$ ) if and only if the inefficiency is sufficiently small in the precise sense that:*

$$\varepsilon \leq \frac{N_0 + 1}{5N_0 + 3}.$$

The remainder of this subsection is concerned with a numerical example, as shown in Figures 3.1 through 3.5; the aim is to communicate the pattern of model results in a different way to the equations used so far.

Figure 3.1 plots the bargaining power of workers,  $K(n)$ , as a function of a fixed union density,  $n/N_0$ , for various elasticities,  $\gamma$ . It shows that a smaller elasticity has a direct positive effect on the workers' bargaining power for any fixed union density.

Figure 3.2 plots the equilibrium bargaining power, union density and wage against the elasticity when the inefficiency is fixed ( $\varepsilon = 2/3$ ). In this case the equilibrium bargaining power is decreasing in the elasticity,



implying that the negative direct effect from the higher elasticity is more than offsetting the positive indirect effect (the positive indirect effect refers to the rising union density, which is also evident on the graph). When the elasticity is one,  $\gamma = 1$ , the union density and workers' bargaining power are the same. This is a result of the specification of the bargaining function; in particular, when  $\gamma = 1$  the bargaining function becomes equal to the union density,  $K(n) = n/N_0$ . The equilibrium wage is falling in the elasticity because of the falling bargaining power.

Figure 3.3 plots the equilibrium wage and union density against the inefficiency for two fixed values of the elasticity ( $\gamma = 1/3$  and  $\gamma = 2/3$ ). As the inefficiency rises, equilibrium wages and the union density fall (the result of Corollary 2). For any given inefficiency, as the elasticity rises from  $\gamma = 1/3$  to  $\gamma = 2/3$ , the equilibrium union density rises and wages fall (as discussed in the description of Figure 3.2).

Figure 3.4 plots the equilibrium total surplus, firm's surplus and workers' surplus. Reflecting Corollary 4, the equilibrium total surplus is unchanged if the inefficiency coefficient rises; with a fall in the workers' surplus exactly offsetting a rise in the firm's surplus. The equilibrium workers' surplus declines as the inefficiency rises because the higher inefficiency reduces the union size and hence the workers' bargaining power. The combination of an unchanged total surplus and the lower bargaining power of workers leads to a rise in the firm's surplus.

Figure 3.5 plots the equilibrium wage and union density against the size of the workforce for two fixed values of the elasticity ( $\gamma = 1/3$  and  $\gamma = 2/3$ ). As the workforce grows, the equilibrium wage increases linearly and the union density is unchanged. In other words, the size of the union grows at the same rate as the workforce in equilibrium. This is largely a result of the workers' bargaining power being a function of union density. For any given workforce size, as the elasticity rises from  $\gamma = 1/3$  to  $\gamma = 2/3$ , the equilibrium wage falls and the union density rises (as discussed in the description of Figure 3.2).

Figure 3.1: Direct Effect from Changes in the Elasticity on Bargaining Power

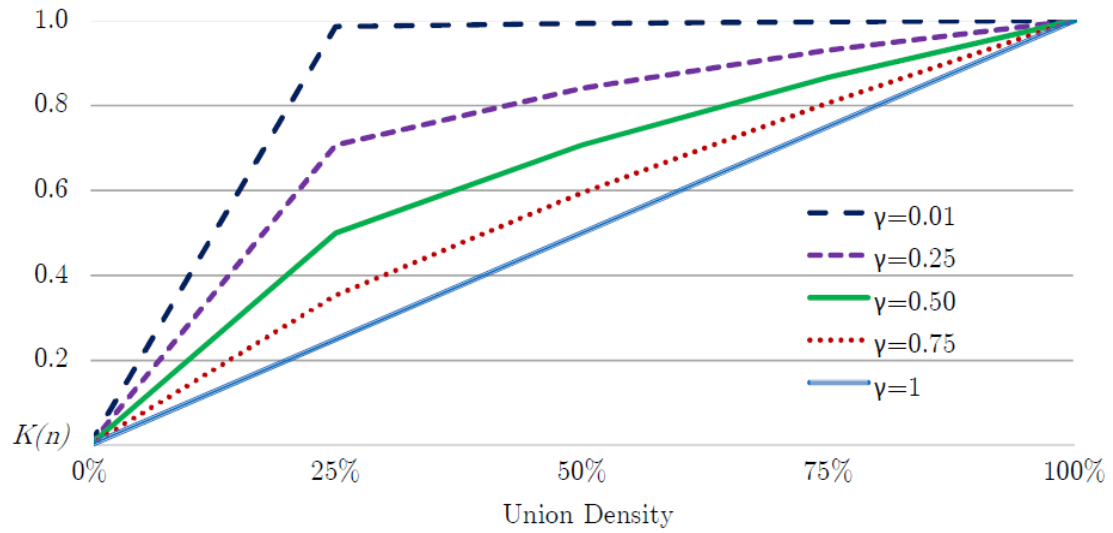


Figure 3.2: Equilibrium Variables as Functions of the Elasticity

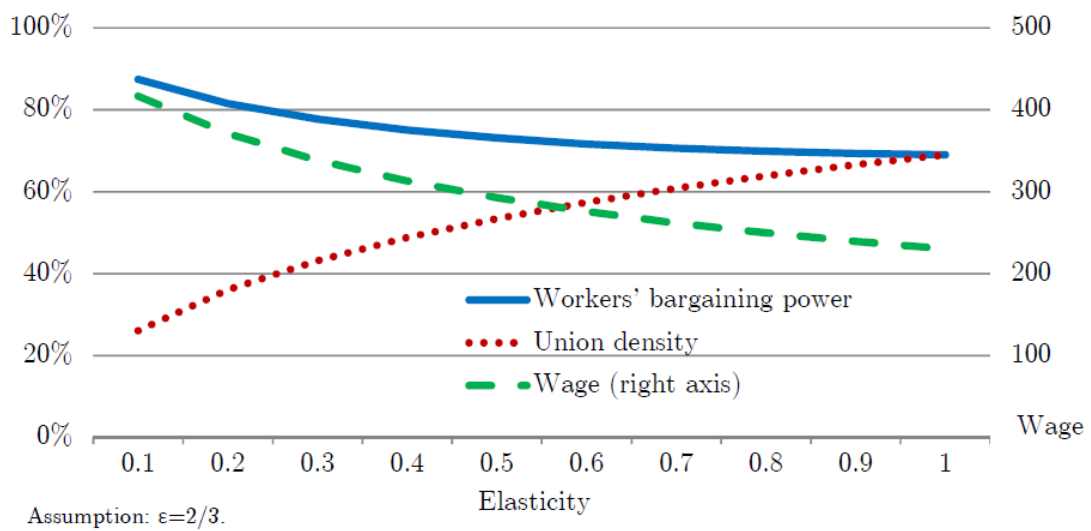


Figure 3.3: Equilibrium Variables as Functions of the Inefficiency Coefficient

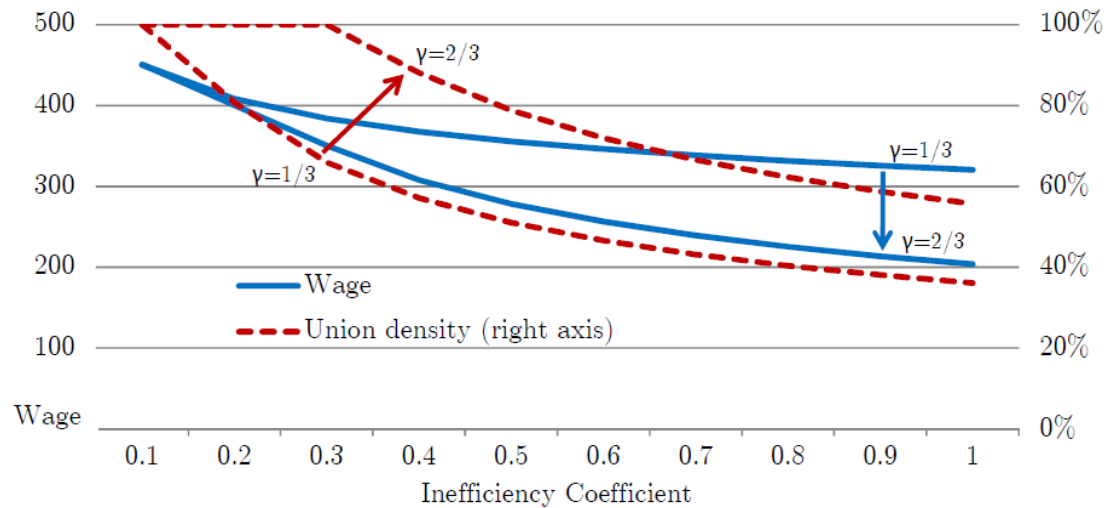


Figure 3.4: Equilibrium Surplus as Functions of the Inefficiency Coefficient

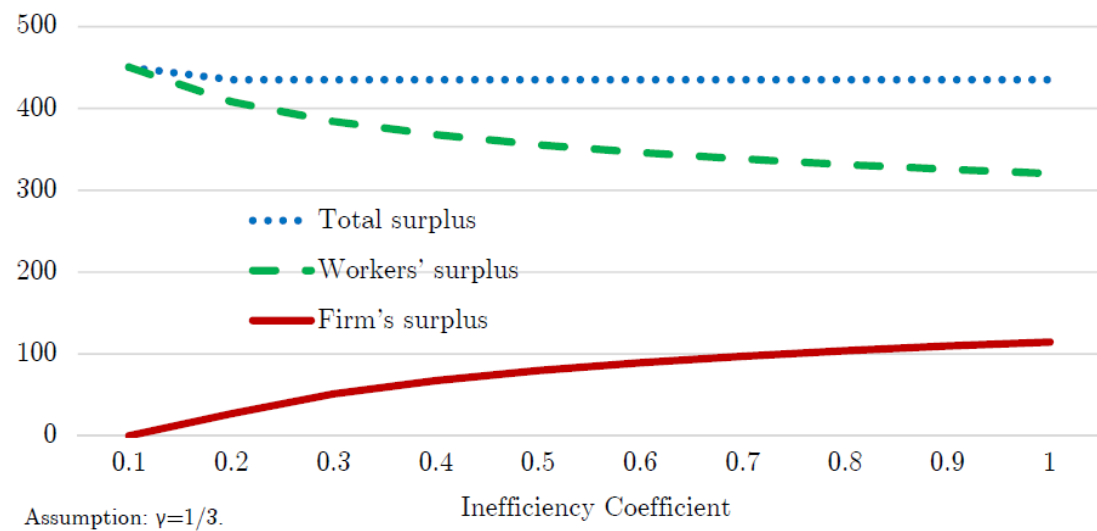
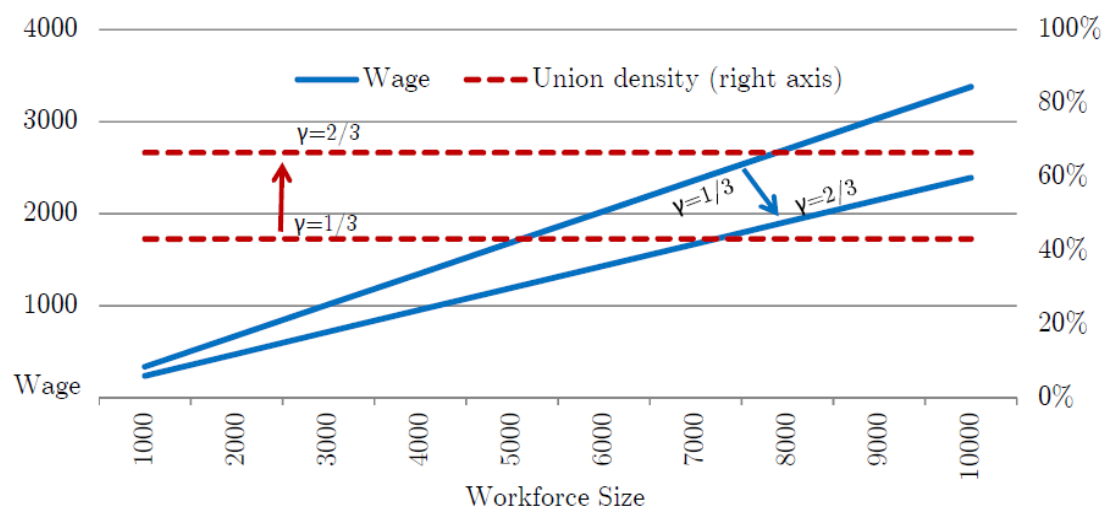


Figure 3.5: Equilibrium Variables as Functions of the Workforce Size



Assumption:  $\varepsilon=2/3$ ,  $\gamma$  rises from  $1/3$  to  $2/3$ .

The next subsection analyzes the case where the workers' bargaining function increases at a constant rate for each new union member; that is, the marginal bargaining power is constant. This would represent the limit case, as  $\gamma \rightarrow 1$ , of this subsection; yet function  $K(n)$  is no longer strictly concave. In such a case, one would expect to observe relatively large unions because the elasticity is relatively high, at least larger than in this example.

### 3.4 Linear Bargaining Power

Consider the class of bargaining power functions that are linearly increasing with the number of union members. Studying this type of bargaining function allows us to examine a case where there is no diminishing marginal return to unionization. Mathematically, there is a constant  $c > 0$  such that the bargaining power is  $K(n) = cn$ , for every  $n \in [0, N_0]$ . This class of bargaining power functions includes the special case where the bargaining power is equal to the union density; that is, the case where  $c = 1/N_0$ , so that  $K(n) = n/N_0$ . Because this function  $K(n)$  violates some of the previous assumptions, equilibrium existence and its properties need to be established separately here. The next result establishes that an equilibrium exists and characterizes this equilibrium.

**Proposition 5** *Suppose  $K(n) = cn$ , for every  $n \in [0, N_0]$ . There exists an interior solution,  $0 < n^* < N_0$ , for the union's problem if and only if the inefficiency coefficient of the union is sufficiently large; specifically, if*

$$\varepsilon > \frac{1}{3} \frac{N_0(N_0 + 1)}{N_0(N_0 + 1) - \frac{1}{3}N_0}.$$

*If the inefficiency coefficient is smaller than or equal to this, then all workers join the union. If there is an interior solution, then the equilibrium number of union members is:*

$$n^* = -\frac{1}{3} + \sqrt{\frac{1}{9} + \frac{N_0(N_0 + 1)}{3\varepsilon}}.$$

As the size of the population of workers grows large,  $N_0 \rightarrow +\infty$ , there is an interior solution if  $\varepsilon > 1/3$ . In this case, the limit of the equilibrium union density, as the size of the population of workers grows large,  $N_0 \rightarrow +\infty$ , is given by:

$$\lim_{N_0 \rightarrow +\infty} \frac{n^*}{N_0} = \frac{1}{\sqrt{3\varepsilon}}. \quad (3.15)$$

From equation (3.15), the equilibrium union density ranges from around 57% when  $\varepsilon = 1$  to 100% when  $\varepsilon = 1/3$ . This is because, by construction, it is favorable for the union to have a high density as there are no diminishing returns from unionization. In contrast, the curvature in the previous sections led to decreasing marginal bargaining power as the size of the union increased. The next corollary is concerned with the comparative statics.

In the limit, it is obvious that the surplus grows to infinity as the most productive worker grows to infinity, but it is not clear what happens to the equilibrium union density and, hence, wages. The next result provides some insight on this. It is concerned with the comparative statistics and, in some relevant cases, the comparative statics in the limit when the workforce size grows large. The way to interpret the limit results is in the context of an industry with a large number of workers; the model at infinity will be an approximation of this.<sup>1</sup>

**Corollary 5** *Suppose  $K(n) = cn$ , for every  $n \in [0, N_0]$ , and  $\varepsilon > 1/3$ , so that the union's problem has an interior solution. In this case, the equilibrium wage and number of union members are decreasing functions of the union inefficiency. Formally:*

$$\frac{d\alpha^*}{d\varepsilon} < 0, \quad \frac{dn^*}{d\varepsilon} < 0.$$

*The equilibrium wage and number of union members are increasing func-*

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<sup>1</sup>It is common in economics literature to have a population of continuous agents.

tions of the total number of workers:

$$\frac{d\alpha^*}{dN_0} > 0, \quad \frac{dn^*}{dN_0} > 0.$$

As  $N_0$  grows large, the equilibrium union density,  $n^*/N_0$ , is a constant. Formally:

$$\lim_{N_0 \rightarrow +\infty} \frac{d(n^*/N_0)}{dN_0} = 0.$$

As  $N_0$  grows large, the equilibrium workers' surplus falls with the inefficiency coefficient, the firm surplus rises, and the total surplus is unchanged. Formally:

$$\lim_{N_0 \rightarrow +\infty} \frac{dY^*}{d\varepsilon} = 0, \quad \lim_{N_0 \rightarrow +\infty} \frac{dY_w^*}{d\varepsilon} < 0, \quad \lim_{N_0 \rightarrow +\infty} \frac{dY_f^*}{d\varepsilon} > 0.$$

As the inefficiency coefficient of the union rises, the equilibrium wage and union density fall. This is because, as the union inefficiency rises, workers in the union face a choice between: (a) staying in the union and becoming less efficient, which reduces the size of the total surplus (that is, the "size of the pie" is reduced); or (b) leaving the union, which reduces their bargaining power (that is, the "share of the pie" is reduced). Under this specification of the workers' bargaining power, it is costlier to stay in the union, hence workers leave the union and wages decline because of the resulting reduction in bargaining power.

As the number of workers rises, so too does the equilibrium number of union members. In equilibrium, the union size rises at the same rate as the workforce size. In other words, the equilibrium union density is independent of the workforce size. The equilibrium wage rises with the size of the workforce, mainly reflecting the increasing surplus as each new worker brings more productivity.

The equilibrium total surplus is unchanged if the inefficiency coefficient rises; with a fall in the workers' surplus exactly offsetting a rise in the firm's surplus. The equilibrium workers' surplus declines as the inefficiency rises

because the higher inefficiency reduces the union size and hence the workers' bargaining power. The combination of an unchanged total surplus and the reduced bargaining power of workers leads to a rise in the firm's surplus.

As in the previous sections, the firm has incentives to increase the inefficiency of union workers, while union workers have incentives to be efficient.

### 3.5 Summary

This chapter proposed the Constant Wage Model and used it to study the strategic interaction between workers and a labor union. The chapter focused on studying the properties of an equilibrium where lower skilled workers joined the labor union while higher skilled workers did not (as discussed earlier, this thesis is concerned with studying the properties of such an equilibrium because labor unions tend to be more popular amongst lower skilled workers (Blanchflower [5])). In the model, the output and skills of workers were unobservable, each worker had a different skill level, and union membership decreased a worker's output, but increased the bargaining power of workers. The model was solved to find expressions for the equilibrium wage, workers' surplus, firm surplus, union size and the union density. Comparative static results were also reported. The main results were:

- An equilibrium exists where lower skilled workers join a labor union and higher skilled workers do not. This equilibrium is unique and robust to unilateral and coalitional deviations. Moreover, no equilibrium exists where higher skilled workers join the union and lower skilled workers do not (there were unilateral and coalitional deviations).
- All workers join the labor union if the cost of doing so (in terms of the inefficiency coefficient) is sufficiently small and there is a benefit from joining (in terms of the workers' bargaining power). The implication is that there can exist fully unionized workforces even when the union decreases the productivity of workers.



- The labor union and the workers always have an incentive to reduce the inefficiency associated with union membership. In particular, labor unions want their members to be as efficient as if they had not joined the union. The inefficiency coefficient always decreases wages and the level of union membership.
- Situations exist where the firm prefers union workers to be less efficient. This is because workers exit the labor union when the inefficiency coefficient increases. The smaller union size associated with the higher inefficiency coefficient reduces the bargaining power of workers and increases the firm's bargaining power.
- These incentives of the firm and union may be important to consider during wage negotiations if the firm and union can influence the inefficiency coefficient.

This chapter also studied the Constant Wage Model with restrictions on the bargaining function. The first restriction was that the bargaining function had a constant elasticity with respect to the union size. The main results under this restriction were that:

- An increase in the elasticity has a negative direct effect on workers' bargaining power and a positive indirect effect (the positive indirect effect is from the larger union size). If labor unions can change the elasticity of their bargaining function they need to consider both effects.
- A higher inefficiency coefficient did not change the total surplus in equilibrium, only the allocation of the surplus. In particular, a higher inefficiency coefficient decreases the equilibrium workers' surplus and increases the firm's surplus. As a consequence, under the constant elasticity bargaining function, the firm always prefers union workers to be inefficient. This is because a higher inefficiency reduces the size

of the union and, hence, workers' bargaining power. This in turn increases the firm's bargaining power.

The second restriction studied the class of bargaining functions that were linearly increasing in the union size. The results under this restriction were similar to the previous restriction and the general case. The main difference was that the equilibrium union size was relatively large because there were no diminishing returns from unionization.

## Chapter 4

# Wages Increase Linearly with Skills

### 4.1 Model Setup

This chapter studies the strategic interaction between workers and a union when the output of workers is observable or verifiable. The model studied in this chapter is almost the same as the model of Chapter 3. Assumptions (3.1) to (3.5) hold and the union inefficiency,  $\varepsilon$ , is exogenously given as in Chapter 3. The only difference is that the output of workers is observable and, hence, workers earn a wage that increases linearly in their productivity. Let  $w(j) = \beta j > 0$  denote the wage for union worker  $j$ . The union bargains with the firm over the rate  $\beta > 0$ . We name the parameter  $\beta$  a wage rate, however, it is equivalent to a return to skill. This is because the wage received by a union worker of skill level  $j$  is the product of  $\beta$  (the wage rate) and  $j$  (their skill). Similarly, non-union workers also earn a wage linearly increasing in their skill in the same fashion as for union workers. The wage rate of non-union workers may be equal to or different from the wage rate of union workers. As described in the following subsection, in equilibrium these wage rates are the same because of the incentives of workers.

The proposed strategy profile is the same as the one described in Chapter

3; workers  $j \in \{1, \dots, n\}$  are union members, and all workers with skills  $j > n$  are not in the union. The Budget Balance constraint of Chapter 3 also holds here; that is, wages in aggregate cannot exceed workers' surplus,  $Y_w$ . If all workers receive the same wage rate  $\beta$  this constraint can be written as:

$$\begin{aligned} Y_w &= \sum_{j=1}^{N_0} \beta j \\ &= \frac{\beta N_0(N_0 + 1)}{2}. \end{aligned} \tag{4.1}$$

Chapter 3 describes the total surplus, the surplus of union workers, and the surplus of non-union workers. Similarly, a strategy profile is defined exactly as in Chapter 3, except with the wage rate,  $\beta$ , the choice variable of the union rather than the constant wage,  $\alpha$ .

## 4.2 Equilibrium

The focus will be on an equilibrium where all workers earn the same wage rate regardless of their unionization status; that is, the wage rate received by non-union workers is the same as the wage rate of union workers,  $\beta$ . There is no equilibrium where the wage rates are not equal because such a scenario would yield profitable deviations for workers receiving the lower wage rate. As described in Chapter 3, the firm chooses to pay non-unionized workers because of the threat effect.

The problem of the union is to maximize the wage rate,  $\beta$ , of its members subject to budget balance.<sup>1</sup> The union anticipates the workers' union membership decisions in the spirit of backward induction. In particular, it knows for each wage rate the corresponding number of workers that will join.

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<sup>1</sup>As described in Chapter 3, the assumption that the union is interested in maximizing the wages of its members is generally not controversial.

Mathematically, the union's problem is:

$$\max_{0 \leq n \leq N_0} \beta \quad \text{subject to equations (3.8) and (4.1).}$$

The next result finds an expression for the wage rate  $\beta$  by combining equations (3.8) and (4.1). The wage rate is a function of the union size, the inefficiency coefficient, the size of the workforce and the bargaining power.

**Lemma 2** *The wage rate,  $\beta$ , in terms of the workers' bargaining power function, inefficiency coefficient and union size is given by:*

$$\beta = K(n) \left[ 1 - \varepsilon \frac{n(n+1)}{N_0(N_0+1)} \right]. \quad (4.2)$$

Let  $\beta^*$  denote the equilibrium wage rate of workers. By equation (4.2), each possible wage rate  $\beta$  is a function of the number of union members,  $n$ . It is straightforward to prove that the function taking  $n$  to  $\beta(n)$  according to equation (4.2) is strictly concave, and  $\beta'(0) > 0$ .

The proposed strategy profile is an equilibrium. Indeed, the union is maximizing its utility by construction. Workers not in the union cannot find a profitable deviation when all other players play their part in the equilibrium; it does not matter if they stay out of the union or not; they always obtain wage rate  $\beta^*$ . As in Chapter 3, the only non-trivial incentive condition is related to the incentives of union workers. If a union member deviates and leaves the union, the bargaining power of workers falls from  $K(n)$  to  $K(n-1)$ , but the total surplus,  $Y$ , rises because the deviating worker's production increases by a fraction  $\varepsilon$ . The next proposition proves that this non-trivial incentive condition is satisfied. Union members do not have incentives to deviate, so the proposed strategy profile is indeed an equilibrium.

**Proposition 6** *Suppose that assumptions (3.1) through (3.5) are satisfied. Then, there exists an equilibrium where the  $n^*$  workers of lowest skill,  $0 \leq n^* \leq N_0$ , join the union, and the  $N_0 - n^*$  workers of highest skill do not*

join the union. There is a unique number of workers in the union,  $n^* \in [0, N_0]$ , solving the union's problem. This unique  $n^*$  is either the upper corner solution  $n^* = N_0$  (all workers are in the union), or it is the interior solution which is implicitly defined by the first order condition:

$$\frac{dK(n^*)}{dn} \frac{n^*}{K(n^*)} = \frac{\varepsilon n^* (2n^* + 1)}{N_0(N_0 + 1) - \varepsilon n^* (n^* + 1)}. \quad (4.3)$$

The equilibrium union size, defined by condition (4.3), is the same as the one given by equation (3.11) in Chapter 3, where skills were unobservable and the wage was constant. Indeed, the observability of skill and the structure of wages does not impact on the equilibrium size of the union. As such, the results of Chapter 3 regarding the equilibrium union size are true in this chapter. In particular, there is an interior solution to the union's problem if the inefficiency coefficient is sufficiently small, at least one worker is always unionized regardless of the union's inefficiency, and the equilibrium union size falls as the inefficiency rises. Moreover, because the union size is the same as in Chapter 3, the equilibrium total surplus, workers' surplus and the bargaining power are also the same as in Chapter 3. The next result establishes how the equilibrium wage rate changes with the inefficiency coefficient.

**Corollary 6** *Suppose assumptions (3.1) through (3.6) are satisfied. In this case, the equilibrium wage rate,  $\beta^*$ , is decreasing in the inefficiency coefficient,  $\varepsilon$ ; that is, as the union inefficiency rises, the equilibrium wage rate falls. Mathematically:*

$$\frac{d\beta^*}{d\varepsilon} < 0.$$

The result of Corollary 6 is implied by Corollary 2, which established that  $d\alpha^*/d\varepsilon < 0$ . This is because  $\beta$  is an increasing transformation of  $\alpha$  (for details, see the proof of Corollary 6).

Because the equilibrium union size is the same as in Chapter 3, and because the wage rate is an increasing transformation of the constant wage

of Chapter 3, the comparative static results of Corollary 2 relating to the equilibrium union size and surplus sizes are also true for this model. In particular, the equilibrium workers' surplus is falling in the inefficiency coefficient.

A key difference between the equilibrium of Chapter 3 (where output was unobservable) and this model (where output is observable) is the size of the workers' surplus accrued by union workers compared to non-union workers. One would expect that when output becomes observable, the more productive non-unionized workers would accrue more in wages, while the relatively less productive unionized workers would accrue less. This is indeed the case, as the next result explains.

**Remark 3** *When output is observable and wages increase with productivity, unionized workers earn, in aggregate, less in equilibrium compared to the case where output is unobservable and all workers earn a constant wage. In other words, union members earn less (in aggregate) in equilibrium in the model setup of this chapter compared to the model setup of Chapter 3. In contrast, non-unionized workers earn more in aggregate in equilibrium when output is observable and wages increase with productivity. The only exception is when all workers are unionized; in this case the amount of the workers' surplus accrued by unionized workers is unchanged between the models.*

While the result of Remark 3 is expected, this comparison depends on the characterization of the equilibrium in each model (in Chapter 3 and Chapter 4). The result suggests that, under the current model setup of observable output, the union has incentives to hide the output of workers or argue against performance-based pay. Indeed, by hiding or reducing the observability of output it can increase the wages of its members. Union campaigns against performance-based pay regularly appear in the media (Buck and Green [15]).

As described earlier, the equilibrium workers' surplus,  $Y_w^*$ , is unchanged between the models of Chapters 3 and 4, and the wage was the same for all

workers in Chapter 3 (where output was unobservable), but increases linearly with skill in Chapter 4 (where output is observable). One implication of these two factors is that the wage received by the worker with the median skill (that is, the worker with skill  $n^M = (N_0 + 1)/2$ ) is the same in both models. Workers with a skill level below that of the median skilled worker (that is, workers with skills  $j < n^M$ ) earn less when output is observable (Chapter 4) compared to when it is unobservable (Chapter 3). In contrast, workers with a skill above that of the median skilled worker (that is, workers with skills  $j > n^M$ ) earn more when output is observable. To put it another way, the relationship between wage and skill level “rotates” around the median skilled worker when output changes from unobservable (constant wage) to observable (wages increase in skill).

If a "large" union exists in equilibrium, in the sense that the marginal union member,  $n^*$ , has a greater skill level than the median skilled worker,  $n^M$ , then some union members (specifically, those with skills between  $n^M$  and  $n^*$ ) will receive a larger wage when output is observable compared to when it is unobservable. However, as Corollary 3 explains, because the union is primarily made up of lower skilled workers, the effect of output observability is to decrease the wages of union workers in aggregate (that is, output observability decreases the workers' surplus accrued by union workers). The only exception is when all workers are unionized: when this is the case, 50% of union workers are worse off and 50% are better off with observable output and the total wage bill (and hence the amount of the surplus accrued by union workers) is unchanged.

### 4.3 Summary

This chapter studied the strategic interaction between workers and a labor union using a similar model to the one presented in Chapter 3; the only difference was that output was observable and wages increased linearly with a worker's productivity. As in Chapter 3, this chapter focused on studying



the properties of an equilibrium where lower skilled workers joined the labor union while higher skilled workers did not.

The equilibrium variables under this model were the same as the Chapter 3 model, where output was observable. In other words, the observability of a worker's output did not change the equilibrium variables. Other main results were that:

- The equilibrium wage rate is decreasing in the inefficiency coefficient.
- Unionized workers generally earn less in aggregate when output is observable compared to when it is unobservable. In other words, unionized workers generally earn less in aggregate in the model setup of this chapter compared to the model setup of Chapter 3. This suggests that unions may have incentives to limit output observability for workers or oppose performance-based pay schemes.

# Chapter 5

## Mixed Wage Schedules

### 5.1 Model Setup

This chapter studies the strategic interaction between workers and a union when the output of non-unionized workers is observable and the output of union workers is unobservable. The model studied in this chapter is almost the same as the model in Chapter 3; the only difference is the assumption on output observability and, hence, the structure of wages. Assumptions (3.1) to (3.5) hold and the union inefficiency,  $\varepsilon$ , is exogenously given.

As in the previous chapters, wages depend on the observability of output. Union members receive a constant wage while non-union workers receive a wage that increases with their skill. Let  $\alpha > 0$  be the wage of each union member, and  $\beta j > 0$  be the wage for non-union worker  $j$ , for a constant  $\beta > 0$  to be determined in equilibrium.

This model is studied to gain insight into union and worker incentives when the union can enable members to hide their output. To gain more understanding of the union's preferences on performance-based pay, this chapter also studies how the equilibrium variables of this model differ from those in Chapters 3 and 4.

The proposed strategy profile is the same as the one described in Chapter 3; workers  $j \in \{1, \dots, n\}$  are union members, and all workers with skills  $j >$

$n$  are not in the union. Also as in Chapter 3,  $G_U$  and  $G_N$  denote, respectively, the total wage bill for all union and non-union workers. Mathematically:

$$\begin{aligned} G_U &= \alpha n, \\ G_N &= \beta \sum_{j=n+1}^{N_0} j = \beta \frac{N_0(N_0 + 1) - n(n + 1)}{2}. \end{aligned}$$

As in Chapter 3, the surplus received by workers is  $Y_w = K(n)Y$  and given by equation (3.8). Also as in Chapter 3, there is budget balance in the sense that the surplus of workers is used to pay wages:  $Y_w = G_U + G_N$ . Therefore:

$$\alpha n = G_U = Y_w - G_N.$$

Equivalently:

$$\alpha n = \frac{K(n)[N_0(N_0 + 1) - \varepsilon n(n + 1)]}{2} - \frac{\beta[N_0(N_0 + 1) - n(n + 1)]}{2}. \quad (5.1)$$

## 5.2 Equilibrium

The focus will be on an equilibrium where the wage of union workers is equal to the highest skilled union member's wage had she not joined the union. There is no equilibrium if this is not the case because of the incentives of workers to choose the highest possible wage. This condition is:

$$\alpha = \beta n. \quad (5.2)$$

To put it another way, if equilibrium condition (5.2) holds, then no non-member  $k$  has an incentive to join as  $\alpha < \beta k$ , and no member  $j$  has an incentive to leave because  $\alpha \geq \beta j$ .

The problem of the union is to maximize the wage payment to each of its members, by choosing their salary  $\alpha$ , subject to budget balance given by equation (5.1) and the incentive condition (5.2). As in Chapters 3 and

4, the union knows for each wage the number of workers that will join. Mathematically, the union's problem is:

$$\max_{\alpha \geq 0} \alpha \quad \text{subject to (5.1) and (5.2).}$$

The next result establishes the wage rate of non-members,  $\beta$ , by combining equations (5.1) and (5.2). The wage rate is a function of the union size, the inefficiency coefficient, the size of the workforce and the bargaining power.

**Lemma 3** *The wage rate of a non-unionized worker in terms of workers' bargaining power, inefficiency coefficient and union size is given by:*

$$\beta = K(n) \frac{N_0(N_0 + 1) - \varepsilon n(n + 1)}{N_0(N_0 + 1) + n(n - 1)}. \quad (5.3)$$

The next proposition proves that the proposed strategy profile is an equilibrium, that the equilibrium is unique, and characterizes the equilibrium.

**Proposition 7** *Suppose that assumptions (3.1) through (3.5) are satisfied. Then, there exists an equilibrium where the  $n^*$  workers of lowest skill,  $0 \leq n^* \leq N_0$ , join the union, and the  $N_0 - n^*$  workers of highest skill do not join the union. There is a unique number of workers in the union,  $n^* \in [0, N_0]$ , solving the union's problem. This unique  $n^*$  is either the upper corner solution  $n^* = N_0$  (all workers are in the union), or it is the unique interior solution, which is implicitly defined by the first order condition:*

$$\frac{d}{dn} (n\beta(n)) = 0 \quad \Leftrightarrow \quad \beta(n) + n \frac{d\beta(n)}{dn} = 0.$$

*Equivalently:*

$$\begin{aligned} \gamma(n^*) = & \frac{\varepsilon n^* \left[ (3n^* + 2)N_0(N_0 + 1) + n^*(n^{*2} - 2n^* - 1) \right]}{[N_0(N_0 + 1) + n^*(n^* - 1)] [N_0(N_0 + 1) - \varepsilon n^*(n^* + 1)]} \\ & - \frac{N_0(N_0 + 1)(N_0(N_0 + 1) - n^{*2})}{[N_0(N_0 + 1) + n^*(n^* - 1)] [N_0(N_0 + 1) - \varepsilon n^*(n^* + 1)]}. \end{aligned} \quad (5.4)$$

*In particular,  $n^* > 0$ ; that is, there are always at least some workers in the union, regardless of the union's inefficiency. Moreover, all workers are unionized ( $n^* = N_0$ ) if and only if the inefficiency is sufficiently small in the precise sense that:*

$$\varepsilon \leq \frac{\gamma(N_0) + \frac{1}{2N_0}}{\gamma(N_0) + \frac{1}{2N_0} + \frac{2N_0+1}{N_0+1}}. \quad (5.5)$$

When not all workers are unionized, equation (5.4) states that the optimal number of union members depends on the elasticity of the bargaining function, the union inefficiency coefficient, and the size of the workforce. As in Chapter 3, if the marginal bargaining power of workers decreases to zero as union membership grows to 100% (that is, if  $\gamma(N_0) = 0$ , or equivalently,  $dK(N_0)/dn = 0$ ) then the unique solution  $n^*$  is interior.

All workers are unionized if and only if the inefficiency coefficient is sufficiently small, as given by inequality (5.5). The right-hand side of this inequality is an increasing function of the elasticity of the workers' bargaining power with respect to the number of unionized workers,  $\gamma(N_0)$ . The larger this elasticity, the more likely full union membership becomes. The right-hand side of inequality (5.5) is a decreasing function of  $N_0$ . This implies that, for any fixed  $\varepsilon$ , as the population of workers increases, it becomes less likely to find full union membership. The next result finds the full union membership condition when the workforce size grows large.

**Corollary 7** *Let  $\gamma_\infty = \lim_{N_0 \rightarrow +\infty} \gamma(N_0)$ . In the limit, as the population of workers grows large,  $N_0 \rightarrow +\infty$ , the full union membership condition (5.5) converges to:*

$$\varepsilon \leq \frac{\gamma_\infty}{\gamma_\infty + 2}.$$

The full membership condition of Corollary 7 is the same as in equation (3.13) of Chapter 3. Indeed, in a large workforce, the full union condition is independent of whether the wage schedule is constant or increasing in skills, it depends only on the elasticity of the bargaining function with respect to the union size. This is because, at full union membership, the models

become very similar. At full membership, all workers earn a constant wage and the union's objective is equivalent to maximizing the workers' surplus – this is the same as in the model of Chapter 3.

The next corollary explains how the equilibrium union size changes when the inefficiency rises in the special case where the elasticity of the bargaining function is constant. Denote  $R_{\alpha\beta}$  as the right-hand side of the equilibrium condition (5.4).

**Corollary 8** *Suppose assumptions (3.1) through (3.6) are satisfied. The equilibrium number of union members,  $n^*$ , is decreasing in the inefficiency coefficient,  $\varepsilon$ , when the following condition holds:*

$$\frac{dn^*}{d\varepsilon} < 0 \quad \Leftrightarrow \quad \frac{\partial\gamma(n^*)}{\partial n} \leq \frac{\partial R_{\alpha\beta}(n^*)}{\partial n}. \quad (5.6)$$

*In other words, the equilibrium union size is always decreasing in the inefficiency coefficient if the elasticity,  $\gamma(n^*)$ , is constant or negative. When the elasticity is positive, the equilibrium union size is decreasing in the inefficiency coefficient so long as the elasticity does not rise too much with union size, as given by the inequality in the right-hand side of condition (5.6).*

The intuition behind Corollary 8 is similar to that of Corollary 2. In particular, when the inefficiency coefficient rises, union workers have a choice between: (a) joining (or remaining) in the union, which reduces their productivity, but increases their bargaining power (or leaves it unchanged); or (b) leaving the union, which increases their productivity, but reduces their bargaining power. Under this model setup, a worker's choice between these two options depends on how the elasticity changes with the union size. For example, if the elasticity increases quickly with union size, the marginal impact on bargaining power from a worker joining the union may be large. In this case, the equilibrium total surplus would be reduced from the higher union size, but the surplus obtained by union workers (the variable to be maximized) may increase because of the larger bargaining power. This ex-

plains why, in some circumstances, a rise in the inefficiency coefficient can increase the equilibrium union size.

As outlined in Chapter 3, the concept of an equilibrium centered on unilateral deviations may not be the most relevant because unions are essentially a collection or coalition of workers. For this reason, the next proposition proves that the equilibrium outlined in Proposition 7 is robust to coalition deviations; that is, no coalition of workers has a profitable deviation from this equilibrium.

**Proposition 8** *Suppose that assumptions (3.1) through (3.5) hold and consider the equilibrium outlined in Proposition 7. In this equilibrium, no coalition of workers has a profitable deviation from the proposed strategy profile.*

As described in Chapter 3, there are three possible types of coalitions: (a) a coalition of entirely non-union workers, (b) a coalition of entirely union workers, and (c) a coalition containing both union and non-union workers. In equilibrium, no coalition that consists entirely of non-union workers has a profitable deviation because each of these workers obtains at least  $\beta^*n^*$  outside the union and  $\alpha^* = \beta^*n^*$  inside the union.

Similarly, no coalition that consists entirely of union workers has a profitable deviation. If union workers leave the union, the impact on wages from lower bargaining power more than offsets the benefit of increased efficiency.

No coalition containing both union and non-union workers has a profitable deviation. If a non-union worker and union worker switch their membership status, the bargaining power remains unchanged, but output falls by  $\varepsilon$  times the absolute value of the difference in skills between these two workers.

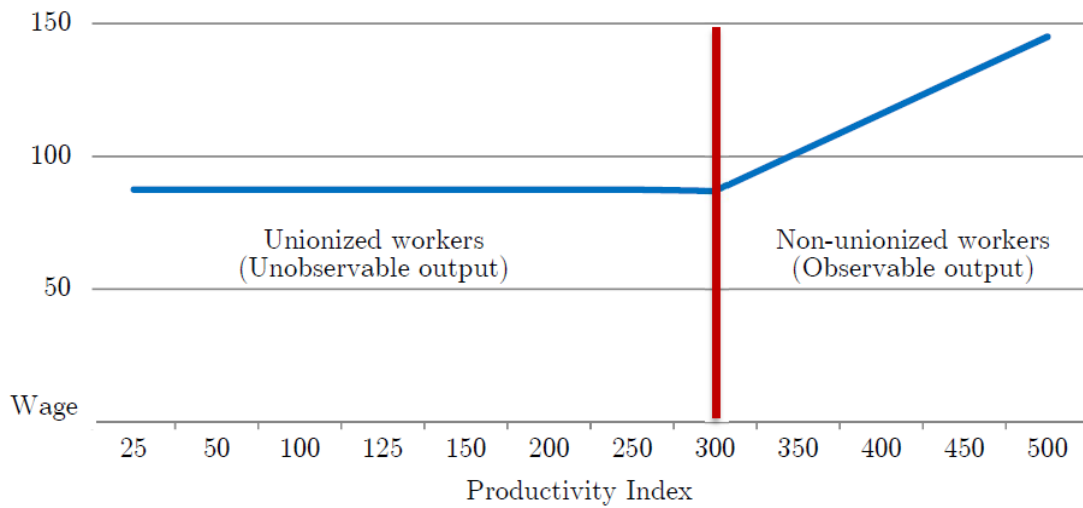
Figure 5.1 plots the equilibrium wage received by workers against their productivity. It shows that the lower skilled workers receive the union wage and the higher skilled workers receive the wage that is increasing in skill.

Figure 5.2 plots the equilibrium wage rate and union density against the size of the workforce. As the size of the workforce grows, the equilibrium

wage rate and union density are unchanged. This implies that the equilibrium number of union workers is rising at the same rate as the workforce and that the wage payments to the most skilled workers are increasing in the workforce size.

Figure 5.3 plots the equilibrium wage rate and union density against the inefficiency. Both variables decline as the inefficiency rises.

Figure 5.1: Equilibrium Wages as a Function of Productivity



Assumptions:  $K(n)=n/N_0$ ,  $\varepsilon=2/3$ .



Figure 5.2: Equilibrium Variables as Functions of the Workforce Size

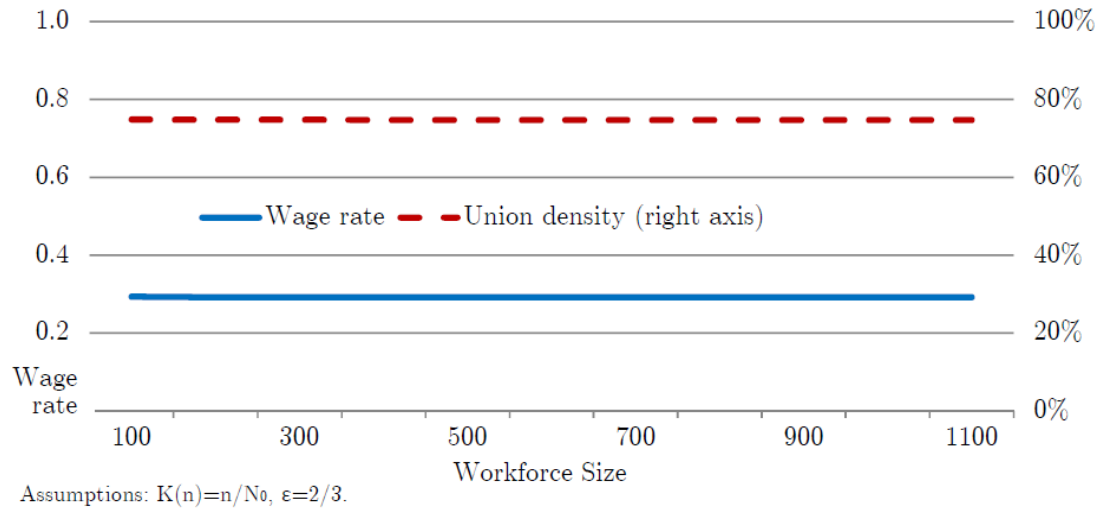
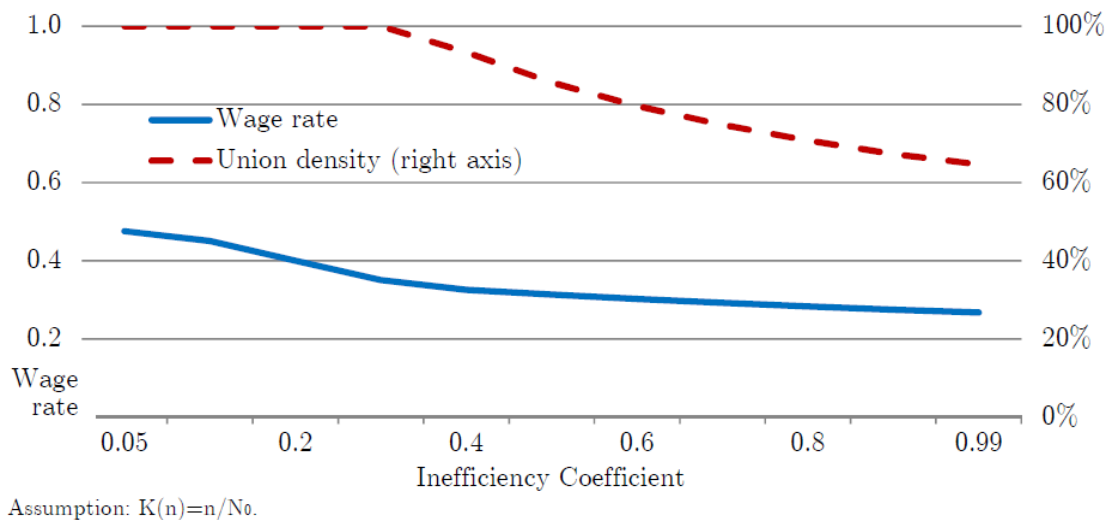


Figure 5.3: Equilibrium Variables as Functions of the Inefficiency Coefficient



### 5.3 Equilibrium Variables Compared to Previous Models

This section compares equilibrium variables across Chapters 3 through 5 to gain insight into how the different assumptions on output observability and the structure of wages affect the strategic interaction between workers and the union.

Let  $R$  be an auxiliary variable that is equal to the absolute value of the elasticity of the total surplus with respect to the union size (that is, it is equal to the right-hand side of condition 3.11 of Chapter 3 except that  $n^*$  is replaced with  $n$ ). Similarly, let  $R_{\alpha\beta}$  be an auxiliary variable representing a negative adjusted elasticity of the total surplus; this auxiliary variable is equal to the right-hand side of condition 5.4 of Chapter 5 except that  $n^*$  is replaced with  $n$ . Mathematically:

$$R = \frac{1}{N_0(N_0 + 1) - \varepsilon n(n + 1)} [\varepsilon n(2n + 1)],$$

$$R_{\alpha\beta} = \frac{\varepsilon n [(3n + 2)N_0(N_0 + 1) + n(n^2 - 2n - 1)]}{[N_0(N_0 + 1) - \varepsilon n(n + 1)][N_0(N_0 + 1) + n(n - 1)]} \quad (5.7)$$

$$- \frac{N_0(N_0 + 1)(N_0(N_0 + 1) - n^2)}{[N_0(N_0 + 1) - \varepsilon n(n + 1)][N_0(N_0 + 1) + n(n - 1)]}.$$

**Remark 4** *The auxiliary function  $R$  is equal to the absolute value of the elasticity of the surplus with respect to the union size; that is,  $R = -E_{Y,n}$ . The auxiliary function  $R_{\alpha\beta}$  is an adjusted or modified elasticity; specifically  $R_{\alpha\beta} = -E_{Y,n} - (N_0(N_0 + 1) - n^2)/(N_0(N_0 + 1) - 1 + n)$ .*

Denote the equilibrium union size in Chapters 3 and 4 as  $n^*$  and the equilibrium union size in Chapter 5 as  $\hat{n}$ . In equilibrium, the auxiliary functions are equal to the elasticity of the bargaining function calculated at their respective union sizes (because of the equilibrium conditions 3.11 and

5.4). Mathematically, in equilibrium:

$$\gamma(n^*) = R(n^*),$$

$$\gamma(\hat{n}) = R_{\alpha\beta}(\hat{n}).$$

The auxiliary functions  $R$  and  $R_{\alpha\beta}$  are increasing with the number of union members, *ceteris paribus*. The next two results establish properties of the auxiliary functions that help us to compare equilibrium variables across the models. The first lemma proves that the auxiliary functions are strictly increasing in the union size,  $n$ .

**Lemma 4** *Fix a "large" workforce size,  $N_0 > 0$ , and any inefficiency coefficient,  $0 < \varepsilon < 1$ . Then, the auxiliary functions  $R(n)$  and  $R_{\alpha\beta}(n)$  are strictly increasing functions of the union size,  $n$ .*

The next result establishes how the auxiliary functions  $R$  and  $R_{\alpha\beta}$  compare with each other for any given union size. In particular, it proves that the auxiliary function  $R$ , for any given workforce size, union size and inefficiency coefficient, is larger than the auxiliary function  $R_{\alpha\beta}$ .

**Lemma 5** *For every inefficiency coefficient  $\varepsilon \in (0, 1)$  and workforce size  $N_0 > 0$ , it turns out that  $R > R_{\alpha\beta}$  holds for every  $n \in [0, N_0]$ .*

To compare the equilibrium union size across models, the elasticity of the bargaining function with respect to union size is assumed to be constant. In particular, there is a constant  $0 < \gamma < 1$  such that  $\gamma(n) = \gamma$ , for every  $n \in [0, N_0]$ . This restriction helps with tractability; assessing the differences between models would be complex and cumbersome for the general cases. As described earlier, the values of the auxiliary functions in equilibrium will be equal to the elasticity:

$$R(n^*) = \gamma,$$

$$R_{\alpha\beta}(\hat{n}) = \gamma.$$

Because the auxiliary functions,  $R(n)$  and  $R_{\alpha\beta}(n)$ , are strictly increasing (by Lemma 4), the function that is lower for any given union size, namely  $R_{\alpha\beta}(n)$  (by Lemma 5), achieves a given elasticity  $\gamma > 0$  at a larger equilibrium union size. In other words, to reach a given constant elasticity, the auxiliary function  $R_{\alpha\beta}(n)$  requires more union members than the auxiliary function  $R(n)$ . Moreover, both functions have inverses, which allows us to write this condition as:

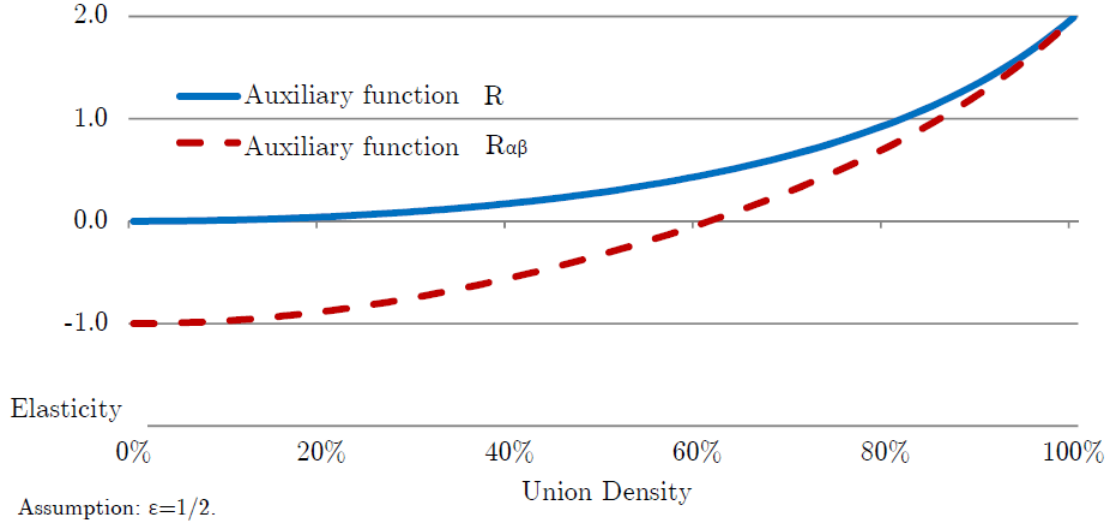
$$\hat{n} = R_{\alpha\beta}^{-1}(\gamma) > R^{-1}(\gamma) = n^*. \quad (5.8)$$

The next result formalizes this idea.

**Proposition 9** *Suppose that the elasticity function  $\gamma(n)$  is constant. Comparing the equilibrium number of union members, the model that pays  $\alpha$  to union members and  $\beta j$  to each non-unionized worker of skill  $j$ , for all  $j > n$ , has more workers in the union, in equilibrium, than the models where all workers have the same type of contract. Formally,  $\hat{n} > n^*$ .*

Figure 5.4 plots the constant elasticity against the equilibrium union density showing this result. In particular, for any given elasticity the auxiliary function  $R(n)$  has a higher union density than the auxiliary function  $R_{\alpha\beta}(n)$ .

Figure 5.4: Elasticity as a Function of the Union Density



An implication of Proposition 9 is that the structure of wages can affect the equilibrium union size. This has implications for the equilibrium surplus, workers' surplus and the workers' bargaining power. In particular, with a higher union size in equilibrium, the model which offers different wage contracts to workers will have more workers affected by the union inefficiency and, hence, have a lower total surplus in equilibrium compared to the model which offers workers the same wage contract. Furthermore, with more union members in equilibrium, the workers in this model must have relatively more bargaining power and, hence, a higher share (in terms of a percentage) of the equilibrium total surplus. For this model, the net impact from the relatively lower total surplus and the relatively higher bargaining power is a lower equilibrium worker surplus compared to the model which offers the same wage contract to all workers. The next corollary formalizes this result. To simplify the English, call the model of Chapter 3 the  $\alpha$ -model, the model of Chapter 4 the  $\beta$ -model and the model of Chapter 5 the  $\alpha\beta$ -model.

**Corollary 9** *Suppose that the elasticity function,  $\gamma(n)$ , is constant. The*

*$\alpha\beta$ -model has a lower total surplus and workers' surplus in equilibrium than the models where all workers have the same type of contract. On the other hand, in the  $\alpha\beta$ -model, workers have more bargaining power and a higher share of the total surplus in equilibrium, compared to the other models. Because the total surplus is lower and the bargaining power of workers is higher, the firm surplus is lower in the equilibrium of the  $\alpha\beta$ -model.*

The implication of Corollary 9 is that, if the union also cares about the size of the workers' surplus, then it prefers the models where all workers face the same wage contract. On the other hand, if the union cares about bargaining power (for example, for political reasons) or the percentage of the surplus that workers obtain, then it prefers the model which offers the two separate wage contracts.<sup>1</sup>

The union may also care about the wage of its members, rather than the wages of workers per se. The next result compares the constant wage received by union workers in Chapter 3 and Chapter 5.

**Corollary 10** *Union workers receive a higher wage, in aggregate, in equilibrium in the model that pays  $\alpha$  to all workers compared to the model which pays  $\alpha$  to union members and  $\beta_j$  to each non-unionized worker of skill  $j$ , for all  $j > n$ .*

The result of Corollary 10 is driven by two factors. First, the workers' surplus is lower in the model that offers two wages contracts compared to the model where all workers receive  $\alpha$  (by Corollary 9). Second, in the model which offers two wage contracts, non-union workers earn more than union workers. In contrast, in the model which offers all workers the same wage, union workers earn the same as non-union workers. Hence, in equilibrium, union workers must earn less in aggregate in the model that offers two wage

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<sup>1</sup>In all models the union is assumed to maximize the wage of union members only; that is, it does not consider other objectives, such as bargaining power. The discussion in this subsection is concerned with what might drive the union's preferences over the model types in this context.

contracts than in the model that offers all workers the same wage. The implication of Corollary 10 is that, if the union's objective is to maximize the wages of its members, it prefers all workers be offered the constant wage. To put another way, it prefers that output for all workers be unobservable.

## 5.4 Summary

This chapter studied the strategic interaction between workers and a labor union using a similar model as in Chapters 3 and 4; the only difference was that output was unobservable for unionized workers and observable for non-unionized workers. As such, wages were constant for unionized workers, but increased with productivity for non-union workers. This setup was designed to reflect situations where a union had successfully opposed performance-based pay for its members. It was designed to study further the union's incentives on performance-based pay. As in Chapter 3, this chapter focused on studying the properties of an equilibrium where lower skilled workers joined the labor union while higher skilled workers did not.

The model was solved to find expressions for the equilibrium wage, the workers' surplus, the firm's surplus, the union size and the union density. Comparative static results were also reported. The equilibrium variables were different from those computed in the previous two chapters. Therefore, this chapter also studied the difference in the equilibrium variables across Chapters 3 through 5. The main results were:

- The proposed strategy profile is an equilibrium. Specifically, there is an equilibrium where unionized workers are lower skilled and earn a constant wage while non-union members are higher skilled and earn a wage that increases linearly in their productivity. This equilibrium is unique and is robust to unilateral and coalitional deviations.
- All workers join the labor union if the cost from doing so (in terms of the inefficiency coefficient) is sufficiently small and there is a benefit from joining (in terms of the workers' bargaining power).

- The equilibrium union size is always decreasing in the inefficiency coefficient if the elasticity of the bargaining function is constant or negative. When the elasticity is positive, the equilibrium union size is decreasing in the inefficiency coefficient so long as the elasticity is not increasing too quickly with the union size.

The difference in the equilibrium variables between the models in Chapters 3 through 5 was studied by restricting the bargaining function. This improved the tractability of the model. The restriction was that the bargaining function had a constant elasticity with respect to the union size. The results shed some light on the implications and possible incentives needed for of the union to choose between the different wage structures. The main results were that:

- The equilibrium size of the union is higher under the model that offered two wage schedules (Chapter 5) compared to the models which offered the same wage contract to all workers (Chapters 3 and 4).
- The equilibrium total surplus, workers' surplus and firm surplus are lower under the model that offered two wage schedules (Chapter 5) compared to the models which offered the same wage contract to all workers (Chapters 3 and 4).
- Union workers earn more in aggregate in the model that offers a constant wage to all workers (Chapter 3) than any of the other models (Chapters 4 or 5). This result held regardless of whether the bargaining function had a constant elasticity or not.



# Chapter 6

## Costly Firm Bargaining with Constant Wages

This chapter studies the interaction between a union and a firm. The model builds on the Constant Wage Model of Chapter 3 by allowing the firm to influence the bargaining power of workers by making costly effort in the bargaining process.

### 6.1 Model Setup

As in Chapter 3, output is unobservable or unverifiable. The union inefficiency,  $\varepsilon$ , is exogenous, with  $0 < \varepsilon \leq 1$ . The strategic players are the union and the firm. The equilibrium concept is the Nash equilibrium of the simultaneous move game of complete information. Fix a workforce size  $N_0 > 0$ . The total surplus,  $Y(n)$ , is defined in the same way as in equation (3.7) of Chapter 3.

Let  $c \geq 0$  denote the amount of costly effort that the firm chooses to use in bargaining with the union. The relative bargaining power of workers is a function  $K : [0, +\infty) \times [0, N_0] \rightarrow \mathbb{R}$ , defined at every pair  $(c, n) \in [0, +\infty) \times [0, N_0]$ . Suppose further that  $K$  is a twice continuously

differentiable function of  $c$  and  $n$  such that:

$$K(c, n) \geq 0 \text{ for all } c \geq 0 \text{ and } 0 \leq n < N_0, \quad (6.1)$$

$$K(c, N_0) \leq 1 \text{ for all } c \geq 0, \quad (6.2)$$

$$\frac{\partial K}{\partial n}(c, n) > 0, \text{ for all } c \geq 0 \text{ and } 0 \leq n < N_0, \quad (6.3)$$

$$\frac{\partial K}{\partial n}(c, N_0) = 0 \text{ for all } c \geq 0, \quad (6.4)$$

$$\frac{\partial K}{\partial c}(c, n) < 0, \text{ for all } c \geq 0 \text{ and all } 0 \leq n \leq N_0, \quad (6.5)$$

$$\frac{\partial^2 K}{\partial n^2}(c, n) \leq 0, \text{ for all } c \geq 0 \text{ and } 0 \leq n < N_0, \quad (6.6)$$

$$\frac{\partial^2 K}{\partial c^2}(c, n) > 0, \text{ for all } c \geq 0 \text{ and all } 0 \leq n < N_0. \quad (6.7)$$

Assumptions (6.1) and (6.2) are similar to assumptions (3.1) and (3.2) of Chapter 3; they ensure the value of the bargaining function is between zero and one. Assumption (6.3) is similar to assumption (3.4) of Chapter 3; it implies that an increase in the union size has a direct positive effect on the workers' bargaining power. Assumption (6.5) implies that an increase in bargaining effort by the firm has a direct negative effect on the workers' bargaining power. This assumption means that more effort that a firm makes in bargaining leads to an increase in the share of the surplus it obtains. Assumptions (6.6) and (6.4) are similar to assumptions (3.5) and (3.6) of Chapter 3. They imply that the bargaining function is weakly concave in the size of the union and that the marginal direct effect from the most skilled worker joining the union is zero. The function  $K : [0, +\infty) \times [0, N_0] \rightarrow \mathbb{R}$  is weakly concave in the union size by assumption (6.6). The function is weakly convex in the bargaining effort by assumption (6.7). This assumption would be expected to hold in reality if there are diminishing marginal returns to the firm's effort.

Define the elasticity of the bargaining function with respect to the size

of the union as  $\gamma_n(c, n)$ . Mathematically:

$$\gamma_n(c, n) = \frac{n}{K(c, n)} \frac{\partial K(c, n)}{\partial n}.$$

To prove the existence and uniqueness of the equilibrium the following assumptions are also required.

$$\frac{\partial K}{\partial c}(0, n) < -\frac{1}{Y(n)} \text{ for all } n \in [0, N_0], \quad (6.8)$$

$$\lim_{c \rightarrow +\infty} \frac{\partial K}{\partial c}(c, n) > -\frac{1}{Y(n)} \text{ for all } n \in [0, N_0], \quad (6.9)$$

$$\frac{\partial K}{\partial n}(c, 0) \geq \frac{\varepsilon K(c, 0)}{N_0(N_0 + 1)} \text{ for all } c \geq 0. \quad (6.10)$$

Assumption (6.8) is similar to assumption (6.5), but builds on it by ensuring there is a minimum marginal direct benefit (in terms of the firm's bargaining power) for the firm to choose a positive non-zero value for its costly effort. Assumption (6.9) is similar to assumption (6.7); it ensures that as the costly effort grows large, the marginal direct benefit from increasing the effort (in terms of the impact on the firm's bargaining power) declines below a certain level. Assumption (6.10) is similar to assumption (6.3), but builds on it by ensuring there is a minimum marginal direct benefit (in terms of the workers' bargaining power) from having at least one worker in the union.

As in the proposed strategy profiles of the previous chapters, workers  $j \in \{1, \dots, n\}$  are union members, and all workers with skills  $j > n$  are not in the union. The surplus of the firm and the workers, respectively, are:

$$\begin{aligned} Y_f(c, n) &= [1 - K(c, n)]Y(n), \\ Y_w(c, n) &= K(c, n)Y(n). \end{aligned} \quad (6.11)$$

The firm chooses the effort,  $c$ , that maximizes its profit, denoted  $\pi$ . The firm's profit is the difference between the firm's surplus and the amount of

costly effort it exerts in the bargaining process, formally:

$$\begin{aligned}\pi(c, n) &= Y_f(c, n) - c \\ &= [1 - K(c, n)]Y(n) - c.\end{aligned}\tag{6.12}$$

## 6.2 Best Reply of the Firm

The firm's best reply is the effort that maximizes its profit given the union's choice of size,  $n$ . Mathematically, the firm's best reply is given by:

$$c^{BR}(n) = \arg \max_{c \geq 0} \pi(c, n).$$

The next proposition proves that the best reply of the firm exists and is unique for every union size. It also characterizes the best reply. It shows that the firm's best reply depends on the surplus, the partial derivative of the bargaining function with respect to  $c$ , and the size of the union.

**Proposition 10** *Suppose assumptions (6.1) through (6.6) hold. The best reply of the firm is the function that takes each  $n \in [0, N_0]$  and returns  $c = c^{BR}(n)$ , the unique positive solution of the first order condition of the firm's profit maximization problem:*

$$-Y(n) \frac{\partial K}{\partial c}(c, n) = 1.\tag{6.13}$$

## 6.3 Wages and the Best Reply of the Union

As in Chapter 3, the amount of wages paid to workers is equal to the workers' surplus,  $Y_w$ . If all workers receive a constant salary  $\alpha$ , then  $Y_w = \alpha N_0$ ; this equality is the same Budget Balance constraint as in Chapter 3 and it is given by condition (3.10).

**Remark 5** *Although we are thinking of workers choosing to unionize, and the union choosing the wage, Chapter 3 showed that this problem is equivalent*

to the union choosing the number of members. In this chapter, we simplify this part of the modelling and assume that the union chooses the number of members,  $n$ .

The problem of the union is to maximize the common wage,  $\alpha$ , by choosing the number of union members,  $n$ , subject to budget balance, for each choice of effort that the firm makes. Mathematically:

$$n^{BR}(c) = \arg \max_{0 \leq n \leq N_0} \alpha(n) \quad \text{subject to } Y_w(c, n) = \alpha N_0.$$

By substituting equation (6.11) for the workers' surplus, the constraint  $\alpha N_0 = Y_w$  becomes:

$$\alpha N_0 = Y_w(c, n) = K(c, n)Y(n) = \frac{K(c, n)}{2} [N_0(N_0 + 1) - \varepsilon n(n + 1)]. \quad (6.14)$$

The wage  $\alpha$  is:

$$\alpha = \frac{K(c, n)}{2N_0} [N_0(N_0 + 1) - \varepsilon n(n + 1)]. \quad (6.15)$$

This wage equation is similar – but not identical – to equation (3.10), which is the wage equation in Chapter 3. The only difference is the bargaining function, which in this chapter depends on the firm's costly bargaining effort,  $c$ , as well as the union size,  $n$ .

The following proposition proves the best reply of the union exists, is unique, and characterizes this best reply.

**Proposition 11** *Suppose assumptions (6.1) through (6.7) hold. In addition, suppose that assumption (6.10) holds. The best reply of the union is the function that takes each  $c > 0$  and returns a unique  $n^{BR}(c)$ . This best reply always exists, and, depending on the parameters, is either the corner solution  $n^{BR}(c) = N_0$  (all workers are in the union), or an interior solution,*

$n^{BR}(c)$ , implicitly defined by the first order condition of the union's problem:

$$\gamma_n(c, n^{BR}) = \frac{\varepsilon n^{BR}(2n^{BR} + 1)}{N_0(N_0 + 1) - \varepsilon n^{BR}(n^{BR} + 1)}. \quad (6.16)$$

Moreover, all workers are unionized ( $n^{BR} = N_0$ ) if and only if the inefficiency coefficient is sufficiently small in the precise sense that:

$$\varepsilon \leq \frac{(N_0 + 1)\gamma_n(c, N_0)}{(N_0 + 1)\gamma_n(c, N_0) + 2N_0 + 1}.$$

The best response of the union is analogous to equation (3.11) defining the union size in Chapters 3 and 4; the only differences are that  $K(c, n)$  replaces  $K(n)$ , and the partial derivative  $\partial K/\partial n$  replaces the derivative  $dK/dn$ . Similarly, the condition for an interior solution to the union's problem is analogous to equation (3.12) of Chapter 3; the only difference is that the partial derivative  $\partial K/\partial n$  replaces the derivative  $dK/dn$ .

The following result helps us prove the uniqueness of the Nash equilibrium. It essentially says that the best reply of one player is increasing (decreasing) at a point where the other player's best reply is decreasing (increasing). Lemma 6 implicitly says that, in any equilibrium, the action of the firm is either a strategic complement (substitute) to the union size, and the union size is a strategic substitute (complement) to the firm effort, in the sense of Bulow [17].

**Lemma 6** *Suppose assumptions (6.1) through (6.7) hold. Then, for all  $n \in [0, N_0]$ :*

$$\frac{dc^{BR}}{dn}(n) = \frac{Y(n)K_{cn}(c^{BR}(n), n) + Y_n(n)K_c(c^{BR}(n), n)}{-Y(n)K_{cc}(c^{BR}(n), n)},$$

and, for all  $c \geq 0$ :

$$\frac{dn^{BR}}{dc}(c) = -\frac{Y(n^{BR}(c))K_{cn}(c, n^{BR}(c)) + Y_n(n^{BR}(c))K_c(c, n^{BR}(c))}{Y_n(n^{BR}(c))K_n(c, n^{BR}(c)) + Y(n^{BR}(c))K_{nn}(c, n^{BR}(c))}.$$

Moreover, suppose that  $(c^*, n^*) \in [0, +\infty) \times [0, N_0]$  is a Nash equilibrium. Then, exactly one of the following three cases is true at  $(c^*, n^*)$ :

$$\begin{aligned} \text{Case 1 : } & \frac{dn^{BR}}{dc}(c^*) > 0, \frac{dc^{BR}}{dn}(n^*) < 0 \Leftrightarrow K_{cn}(c^*, n^*) > \frac{-Y_n(n^*)K_c(c^*, n^*)}{Y(n^*)} \\ \text{Case 2 : } & \frac{dn^{BR}}{dc}(c^*) < 0, \frac{dc^{BR}}{dn}(n^*) > 0 \Leftrightarrow K_{cn}(c^*, n^*) < \frac{-Y_n(n^*)K_c(c^*, n^*)}{Y(n^*)} \\ \text{Case 3 : } & \frac{dn^{BR}}{dc}(c^*) = 0, \frac{dc^{BR}}{dn}(n^*) = 0 \Leftrightarrow K_{cn}(c^*, n^*) = \frac{-Y_n(n^*)K_c(c^*, n^*)}{Y(n^*)}. \end{aligned}$$

In all cases:

$$\frac{dn^{BR}}{dc}(c^*) \frac{dc^{BR}}{dn}(n^*) \leq 0.$$

The equilibrium conditions given by Lemma 6 imply that the ‘stability condition’ utilized in Mas-Colell, Whinston and Green [38] is satisfied in any equilibrium (see Mas-Colell, Whinston and Green [38], page 414). This condition can be described as follows. Suppose the players are playing a strategy profile that is ‘close’ to an equilibrium and consider the following dynamic adjustment process. Start at any point  $(c_0, n_0)$  that is extremely close to  $(c^*, n^*)$ , but  $c_0 \neq c^*$  and  $n_0 \neq n^*$ . Player 1 (the firm) at  $t = 1$  plays a best reply against the action of Player 2 (the union). At  $t = 2$ , Player 2 plays their best response to the action of Player 1. For every  $k \in \{1, 2, 3, \dots\}$ , at  $t = 2k + 1$ , Player 1 plays their best reply against Player 2 in period  $2k$ . At  $t = 2k + 2$ , Player 2 plays a best reply against the action of Player 1 in period  $t = 2k + 1$ . The stability condition requires that this dynamic adjustment process converges to the Nash equilibrium.

Consider the following Jacobian matrix.

$$\begin{pmatrix} \alpha_{nn} & \alpha_{nc} \\ \pi_{nc} & \pi_{cc} \end{pmatrix}$$

The stability condition requires that the Jacobian determinant is positive, or equivalently  $\alpha_{nn}\pi_{cc} - \alpha_{nc}\pi_{cn} > 0$ . In our case, it is clear that  $\alpha_{nn} > 0$  and  $\pi_{cc} > 0$ , but the signs of  $\alpha_{nc}$  and  $\pi_{cn}$  depend on  $K_{cn}$  (Appendix C con-

tains details of these partial derivatives). In equilibrium, one of the partial cross derivatives,  $\pi_{cn}$  or  $\alpha_{nc}$ , is negative and one is positive, or they are both zero. In fact, in equilibrium this is the case:

$$\begin{aligned} \text{Case 1 : } \quad K_{cn}(c^*, n^*) &> \frac{-Y_n(n^*)K_c(c^*, n^*)}{Y(n^*)} &\Leftrightarrow \quad \alpha_{nc} > 0, \pi_{cn} < 0 \\ \text{Case 2 : } \quad K_{cn}(c^*, n^*) &< \frac{-Y_n(n^*)K_c(c^*, n^*)}{Y(n^*)} &\Leftrightarrow \quad \alpha_{nc} < 0, \pi_{cn} > 0 \\ \text{Case 3 : } \quad K_{cn}(c^*, n^*) &= \frac{-Y_n(n^*)K_c(c^*, n^*)}{Y(n^*)} &\Leftrightarrow \quad \alpha_{nc} = 0, \pi_{cn} = 0. \end{aligned}$$

In all cases  $\alpha_{nc}\pi_{cn} \leq 0$ . Hence,  $\alpha_{nn}\pi_{cc} - \alpha_{nc}\pi_{cn} > 0$ . The three cases in Lemma 6 imply that, in any equilibrium, the determinant of the Jacobian is positive and, hence, the stability condition must hold. Essentially, Lemma 6 says that there is no equilibrium where the actions of both players are strategic substitutes, or where the actions of both players are strategic complements. Appendix C calculates the partial derivatives,  $\alpha_{nn}$ ,  $\pi_{cc}$ ,  $\alpha_{nc}$  and  $\pi_{cn}$ , and shows that the stability condition holds in any equilibrium.

## 6.4 Nash Equilibrium

A Nash equilibrium is the pair  $(c^*, n^*)$ , where  $c^*$  is the best reply of the firm when the union plays  $n^*$ , and  $n^*$  is a best reply of the union when the firm plays  $c^*$ . The next proposition proves that a unique Nash equilibrium exists and is unique.

**Proposition 12** *Suppose assumptions (6.1) through (6.10) hold. Then, a Nash equilibrium  $(c^*, n^*)$  exists and is unique. The equilibrium  $(c^*, n^*)$  solves the following system of equations:*

$$\left\{ \begin{array}{l} \gamma_n(c^*, n^*) = \frac{\varepsilon Y_n(n^*)}{Y(n^*)} \\ -\frac{\partial K}{\partial c}(c^*, n^*) = \frac{1}{Y(n^*)}. \end{array} \right.$$



**Remark 6** *A standard result is that a Nash equilibrium is unique when the derivatives of the best reply functions have opposite signs across their whole domain. Proposition 12 does not use this standard result because we cannot control the signs of these derivatives over their entire domain. Proposition 12 proves a stronger result; one in which we start with local conditions to find a global result. This is why we needed to obtain the stability of every Nash equilibrium in Lemma 6. To put it another way, in the proof of Proposition 12 it is necessary to show that every Nash equilibrium will be stable. The proof of Proposition 12 first establishes existence, and then, argues that each Nash equilibrium is stable. Finally, the proof makes an argument of why only one Nash equilibrium can exist in this situation. See Part II of the proof of Proposition 12 for more detail.*

The following result explains the relationship between the inefficiency coefficient and the marginal effect of the firm's effort on profit. It also explains the relationship between the inefficiency coefficient and the marginal effect of the union size on the wage.

**Corollary 11** *Suppose assumptions (6.1) through (6.10) hold. In this case, an increase in the inefficiency coefficient reduces the direct marginal effect of union membership on the equilibrium wage. Similarly, an increase in the inefficiency coefficient reduces the direct marginal effect of the firm's effort on equilibrium profit. Mathematically:*

$$\alpha_{n\epsilon}^* < 0$$

$$\pi_{c\epsilon}^* < 0.$$

So far it has been assumed that the bargaining function is generic. This made derivation of explicit equilibrium conditions and properties difficult. The following subsection overcomes this limitation to some degree by restricting the bargaining function. This allows us to gain additional insight into the strategic interaction compared to the case when the bargaining function is generic.

## 6.5 Example: Particular Bargaining Function

Suppose in this example that the bargaining function is a ratio of the union size to the firm's costly effort. In particular, the relative bargaining power of workers is the function  $K : [0, +\infty) \times [0, N_0] \rightarrow \mathbb{R}$ , defined at every pair  $(c, n) \in [0, +\infty) \times [0, N_0]$  by:

$$K(c, n) = \frac{n^2}{c + n^2}. \quad (6.17)$$

This functional form was chosen for a number of reasons. First, this specification represents the ratio of the firm and union effort; that is, it allows for the (realistic) situation where an increase in the firm's effort increases its bargaining power, and an increase in unionization increases the bargaining power of workers. Second, the specification allows for all variables to be of the same unit; that is, the function allows for the firm's profit, surplus and bargaining cost to be in terms of persons squared. Indeed, equation (6.12) defining the firm's profit suggests these variables should have the same unit, and equation (3.8) defining the total surplus suggests this unit should be persons squared (because the surplus is in terms of persons squared). Third, this specification has an ambiguous sign of the cross derivative  $K_{cn}$ , which makes it more comparable to the general case studied in the previous section where  $K_{cn}$  was also ambiguous.<sup>1</sup>

The bargaining function,  $K$ , is decreasing convexly with the bargaining cost,  $c$ , and increasing with the union size,  $n$ . The sign of the cross partial derivative of the bargaining function (with respect to the firm's costly effort and the union size) depends on the relative magnitudes of the firm's costly

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<sup>1</sup>In practice, the particular choice of specification is not of any practical relevance for this subsection, so long as it behaves in a similar way to the general function studied in the previous section. To put it another way, any specification of  $K$  with a reasonable justification would have served the purpose for this subsection. This is because the point of this subsection is to gain intuition by deriving closed form analytic solutions and numeric results to communicate the pattern of results in a different way to the equations of the previous section; this would not be expected to depend much on the exact choice amongst functional forms with the same or very similar characteristics.

effort and the union size. Mathematically, the properties of this bargaining function are:

$$\frac{\partial K}{\partial c} = \frac{-n^2}{(c + n^2)^2} < 0, \quad (6.18)$$

$$\frac{\partial^2 K}{\partial c^2} = \frac{2n^2}{(c + n^2)^3} > 0, \quad (6.19)$$

$$\frac{\partial K}{\partial n} = \frac{2nc}{(c + n^2)^2} > 0, \quad (6.20)$$

$$\frac{\partial^2 K}{\partial n^2} = \frac{2c(c - 3n^2)}{(c + n^2)^3}, \quad (6.21)$$

$$\frac{\partial^2 K}{\partial c \partial n} = \frac{2n(n^2 - c)}{(c + n^2)^3}, \quad (6.22)$$

$$\gamma_n = \frac{n}{K} \frac{\partial K}{\partial n} = \frac{2c}{c + n^2} > 0. \quad (6.23)$$

Equations (6.18), (6.19) and (6.20) are comparable to assumptions (6.5), (6.7) and (6.3) of the previous section, respectively. In particular, the signs of these partial derivatives and the partial second order derivative are the same as those assumed in the previous section. The economic intuition for these properties is provided in the previous section. Equation (6.21) implies the sign of the second order partial derivative of the bargaining function with respect to union size depends on the relative size of the union and the firm's costly effort. This is different from assumption (6.6) of the previous section, which implied this second order partial derivative was no greater than zero.

The remainder of this subsection calculates the best responses of the union and firm. The equilibrium characteristics and comparative statics are also computed and analyzed. The following proposition outlines the firm's optimal choice of bargaining effort.

**Proposition 13** *Suppose the bargaining function is given by equation (6.17). The best reply of the firm is the function that takes each  $n \geq 0$  and returns  $c^{BR}(n)$ , the unique positive solution of the firm's first order condition, given*

by:

$$c^{BR}(n) = n \sqrt{\frac{N_0(N_0 + 1) - \varepsilon n(n + 1)}{2}} - n^2. \quad (6.24)$$

The best reply of the firm is decreasing in the union size if the cross derivative  $K_{cn}$  is positive or, equivalently, if  $n^2 > c$ . Formally:

$$\frac{dc^{BR}}{dn} < 0 \quad \Leftrightarrow \quad n^2 > c.$$

Equation (6.24) characterizes the best reply of the firm against any choice  $n$  of the union. The optimal bargaining cost depends on the size of the workforce, the number of union members and the inefficiency coefficient.

The next result finds an expression for the wage by using the bargaining power function, the firm's best response function and the budget constraint.

**Lemma 7** *Suppose the bargaining function is given by equation (6.17). Substituting  $c = c^{BR}(n)$  of equation (6.24) and the bargaining power function into equation (6.15) for the wage results in:*

$$\alpha = \frac{\sqrt{2}}{2} \frac{n}{N_0} \sqrt{N_0(N_0 + 1) - \varepsilon n(n + 1)}. \quad (6.25)$$

As in Chapter 3, the wage depends on the size of the workforce, the number of union members and the inefficiency coefficient.

Define  $c_\infty^*$  as the ratio of the firm's costly effort in equilibrium (in terms of persons squared) to the size of the workforce squared when the workforce size is large. Mathematically:

$$c_\infty^* = \lim_{N_0 \rightarrow +\infty} \frac{c^*}{N_0^2}.$$

The variable  $c_\infty^*$  is a finite real number. In particular, divide equation (6.24) by  $N_0^2$  and let  $n = n^*$  and  $c^{BR} = c^*$ , yielding:

$$\frac{c^*}{N_0^2} = \frac{n^*}{N_0} \sqrt{\frac{1}{2} \left( \frac{N_0(N_0 + 1)}{N_0^2} - \frac{\varepsilon n^*(n^* + 1)}{N_0^2} \right)} - \frac{n^{*2}}{N_0^2}.$$

Define  $\eta_\infty^* = \lim_{N_0 \rightarrow +\infty} (n^*/N_0)$  as the equilibrium union density when the workforce size is large. Taking the limit as the workforce size grows large,  $N_0 \rightarrow +\infty$ , on both sides of the above equation results in:

$$c_\infty^* = \frac{\eta_\infty^*}{\sqrt{2}} \sqrt{1 - \varepsilon \eta_\infty^{*2} - \eta_\infty^{*2}}.$$

Because the equilibrium union density is between zero and one, the variable  $c_\infty^*$  is clearly a finite real number.

The following proposition proves the best reply of the union exists and characterizes this best reply.

**Proposition 14** *Suppose the bargaining function is given by equation (6.17). The best reply of the union is the function that takes each  $c > 0$  and returns a unique value of  $n^{BR}(c)$ . Depending on the parameters, the best reply is either  $n^{BR} = N_0$  (all workers are in the union) or the unique interior solution implicitly defined by the first order condition of the union's problem, given by:*

$$n^{BR}[2n^{BR3} + n^{BR2} + 4cn^{BR} + 3c] - \frac{2cN_0(N_0 + 1)}{\varepsilon} = 0. \quad (6.26)$$

*As the size of the workforce grows, full membership occurs if and only if the inefficiency,  $\varepsilon$ , is sufficiently small, in the precise sense that:*

$$\varepsilon \leq \frac{c_\infty^*}{1 + 2c_\infty^*}.$$

*The best reply of the union is increasing in the firm's costly effort:*

$$\frac{dn^{BR}}{dc} > 0.$$

Equation (6.26) shows that the optimal number of union members depends on the size of the workforce, the bargaining effort of the firm and the inefficiency coefficient.

The next proposition proves that a Nash equilibrium of the game exists and characterizes this equilibrium.

**Proposition 15** *Suppose the bargaining function is given by equation (6.17). Then, a Nash equilibrium  $(c^*, n^*)$  exists and is unique when  $n^2 > c$ . The equilibrium  $(c^*, n^*)$  solves the following system:*

$$\begin{cases} c^* = n^* \sqrt{\frac{N_0(N_0+1) - \varepsilon n^*(n^*+1)}{2}} - n^{*2} \\ n^*[2n^{*3} + n^{*2} + 4c^*n^* + 3c^*] = \frac{2c^*N_0(N_0+1)}{\varepsilon}. \end{cases}$$

The system of equations in Proposition 15 are the two best replies given by equations (6.24) and (6.26), but with  $n^*$  replacing  $n$  and  $c^*$  replacing  $c$ . This system does not appear to have a simple analytical solution. However, a solution may be calculated in the limit, as the number of workers grow large,  $N_0 \rightarrow +\infty$ . Recall that  $\eta_\infty^* = \lim_{N_0 \rightarrow +\infty} (n^*/N_0)$ . The next proposition computes the best replies of the firm and union and finds the equilibrium union density as the workforce size grows large,  $N_0 \rightarrow +\infty$ .

**Proposition 16** *Suppose the bargaining function is given by equation (6.17) and that  $N_0 \rightarrow +\infty$ . Further suppose that  $\varepsilon > 0$ . The Nash equilibrium  $(c^*, n^*)$  exists, is unique, and solves the system:*

$$c_\infty^* = \frac{\eta_\infty^{*4}}{\varepsilon^{-1} - 2\eta_\infty^{*2}}, \quad (6.27)$$

$$c_\infty^* = \frac{\eta_\infty^*}{\sqrt{2}} \sqrt{1 - \varepsilon \eta_\infty^{*2}} - \eta_\infty^{*2}. \quad (6.28)$$

*The firm's choice of costly effort is always positive in this equilibrium. The union density is also positive and given by:*

$$\eta_\infty^* = \left(1 + 2\varepsilon + \sqrt{1 + 2\varepsilon}\right)^{-1/2}. \quad (6.29)$$

*This equilibrium always leads to an interior solution to the union's prob-*

lem; that is,  $n^* < N_0$ .

Proposition 16 shows that, as the number of workers grows large, the equilibrium union density and best replies of the firm and union depend only on the exogenously given inefficiency coefficient. It is clear from equation (6.29) that the equilibrium union density is declining in the inefficiency coefficient. Proposition 16 also shows that complete unionization of the workforce never occurs.

Figure 6.1 plots the equilibrium wage, union density and workers' bargaining power against the inefficiency coefficient. It shows that these variables are decreasing in the inefficiency. This is because, as the inefficiency rises, union workers produce less (a direct negative effect on the total surplus and wage). Some workers leave the union when the inefficiency coefficient rises so they can produce more efficiently (as shown by the declining equilibrium union density and bargaining power; a positive indirect effect on the total surplus and wage). The net impact from these two countering effects is to reduce the equilibrium wage.

Figure 6.2 plots the equilibrium workers' surplus, firm surplus and firm's bargaining effort against the inefficiency coefficient. It shows that the equilibrium workers' surplus is decreasing in the inefficiency coefficient (for the same reason the equilibrium wage is declining in the inefficiency coefficient, as described for Figure 6.1). It also shows the firm's equilibrium bargaining effort, surplus and profit are increasing in the inefficiency coefficient. The optimal bargaining effort is increasing in the inefficiency coefficient because the higher inefficiency reduces the equilibrium union size and this increases the marginal profit from bargaining. The increase in the firm's relative bargaining power leads to an increase in the firm's equilibrium surplus and profit.

Figure 6.1: Equilibrium Variables as Functions of the Inefficiency Coefficient

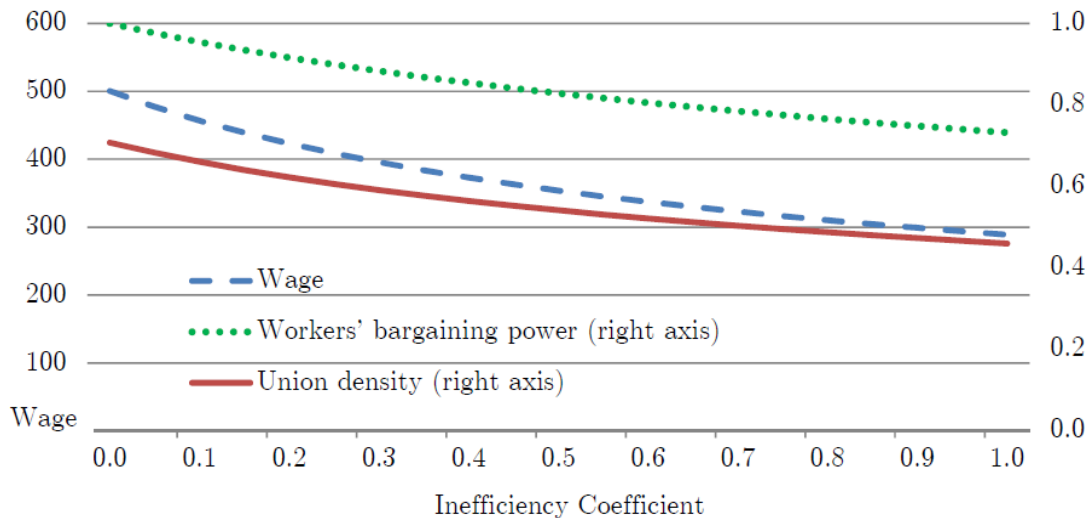
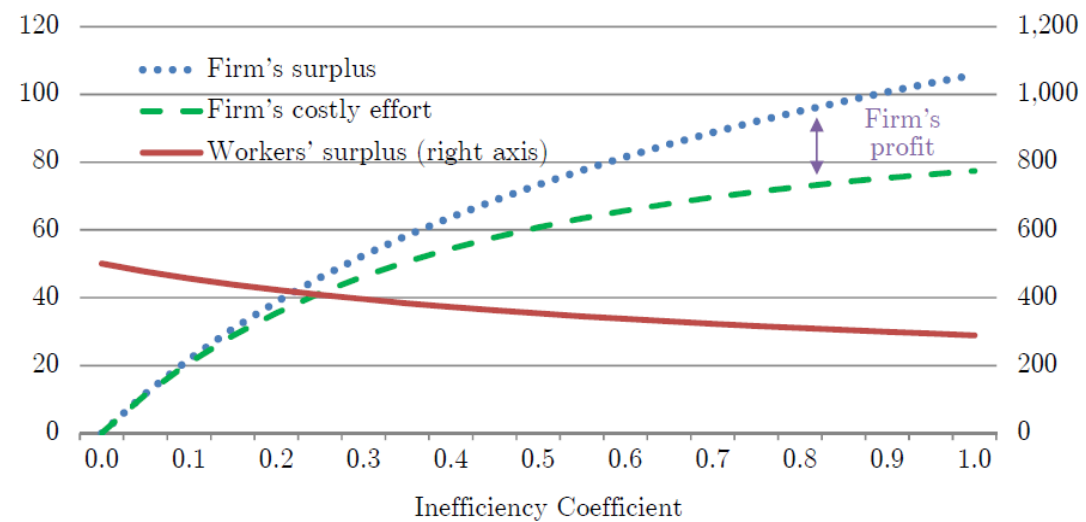


Figure 6.2: Equilibrium Surplus as Functions of the Inefficiency Coefficient





The next result explains how equilibrium variables respond to changes in the inefficiency coefficient and the size of the workforce when the elasticity of the bargaining function with respect to union size is constant. As in Chapter 3, we restrict our attention to the constant elasticity case to improve tractability.

**Corollary 12** *Suppose the bargaining function is given by equation (6.17) and that the bargaining elasticity,  $\gamma_n(c, n)$ , is constant. In this case, the equilibrium number of union members and the firm's bargaining effort are decreasing functions of the inefficiency coefficient. Formally:*

$$\frac{dn^*}{d\varepsilon} < 0, \quad \frac{dc^*}{d\varepsilon} < 0.$$

*As the workforce size,  $N_0$ , grows large, the equilibrium wage is falling in the inefficiency coefficient if and only if the elasticity is sufficiently small. Mathematically:*

$$\lim_{N_0 \rightarrow +\infty} \frac{d\alpha^*}{d\varepsilon} < 0 \quad \Leftrightarrow \quad \gamma_n < \frac{2}{\varepsilon} \left( 1 + \varepsilon - \sqrt{1 + 2\varepsilon} \right).$$

*The equilibrium wage, number of union members and firm effort are increasing functions of the total number of workers:*

$$\frac{d\alpha^*}{dN_0} > 0, \quad \frac{dn^*}{dN_0} > 0, \quad \frac{dc^*}{dN_0} > 0.$$

*In equilibrium, and as the workforce size grows large, the union density is constant. Formally:*

$$\lim_{N_0 \rightarrow +\infty} \frac{d(n^*/N_0)}{dN_0} = 0.$$

*As the size of the workforce grows large, the equilibrium total surplus, workers' surplus and firm surplus fall as the inefficiency coefficient rises if and*

only if the elasticity  $\gamma_n$  is sufficiently small. Formally:

$$\left. \begin{aligned} \lim_{N_0 \rightarrow +\infty} \frac{dY^*}{d\varepsilon} &< 0 \\ \lim_{N_0 \rightarrow +\infty} \frac{dY_f^*}{d\varepsilon} &< 0 \\ \lim_{N_0 \rightarrow +\infty} \frac{dY_w^*}{d\varepsilon} &< 0 \end{aligned} \right\} \Leftrightarrow \gamma_n < \frac{2}{\varepsilon} \left( 1 + \varepsilon - \sqrt{1 + 2\varepsilon} \right).$$

As the size of the workforce grows large, the equilibrium profit of the firm falls as the inefficiency coefficient rises if and only if the elasticity  $\gamma_n$  is sufficiently small. Formally:

$$\lim_{N_0 \rightarrow +\infty} \frac{d\pi^*}{d\varepsilon} < 0 \Leftrightarrow \frac{\varepsilon\gamma_n}{2} + \frac{\gamma_n^2}{(2 - \gamma_n)(1 - K(c^*, n^*)) (1 + 2\gamma_n)} < \varepsilon^2 \sqrt{1 + 2\varepsilon + \sqrt{1 + 2\varepsilon}}.$$

The result that the equilibrium union size is falling in the inefficiency coefficient is similar to the results of previous chapters (for example, Corollary 2 of Chapter 3). As described in the earlier chapters, workers leave the union when the union inefficiency rises because it is essentially more profitable for workers to reduce their bargaining power and improve (aggregate) efficiency.

Corollary 12 showed that when the bargaining function has a constant elasticity, the optimal bargaining effort of the firm is declining in the inefficiency coefficient. This is because the higher inefficiency reduces the equilibrium union size and this increases the relative bargaining power of the firm. In response to this higher bargaining power, the firm reduces its bargaining effort.

Corollary 12 proves that an increase in the inefficiency coefficient can increase or decrease the equilibrium wage, surplus and profit depending on the relative magnitudes of the inefficiency coefficient and the elasticity  $\gamma_n$ . For example, a rise in the inefficiency coefficient leads to a reduction in the

union size. If the elasticity is sufficiently large, this puts sufficient upward pressure on the firm's bargaining power such that the firm's best response is to reduce its costly effort. This in turn can increase the firm's profit.

The comparative static results on the equilibrium union density and wage are similar to the results of Corollary 5 of Chapter 3.

Corollary 12 also shows that the firm's optimal bargaining effort is increasing in the workforce size. This is because a larger workforce size increases the equilibrium size and bargaining power of the union. In response to this reduction in its relative bargaining power, the firm increases its bargaining effort.

## 6.6 Summary

This chapter studied the interaction between workers, a union and a firm using a similar model as Chapter 3; the only difference was that the firm could influence the bargaining power of workers by making costly effort in the bargaining process. As in Chapter 3, output was unobservable and the focus was on the properties of an equilibrium where lower skilled workers joined the labor union while higher skilled workers did not.

The model was solved to find expressions for the equilibrium wage, the workers' surplus, the firm's profit, the firm's bargaining effort, the union size and the union density. Comparative static results were also reported. The main results were:

- The best replies of the firm and union exist and are unique.
- All workers join the labor union if the cost from doing so (in terms of the inefficiency coefficient) is sufficiently small and there is a benefit from joining (in terms of the workers' bargaining power). In some cases, the union had to improve the productivity of its members before all workers would unionize.

- A unique equilibrium exists where the union and firm play their best responses.

This chapter also studied the model under a restriction on the bargaining function. This improved the tractability of the model and allowed equilibrium variables to be expressed in terms of the exogenous parameters only. The restriction was that the bargaining function was a ratio of the union size to the firm's bargaining effort. The main results under this restriction were that:

- A higher inefficiency coefficient decreases the equilibrium union size and the firm's bargaining effort. The equilibrium union size is decreasing in the inefficiency coefficient because a higher inefficiency effectively increases the cost of unionization. The firm's optimal bargaining effort is decreasing in the inefficiency coefficient because the union size, and hence workers' bargaining power, decreases.
- A higher inefficiency coefficient decreases the equilibrium firm profit if the elasticity of the bargaining function with respect to union size is not too large. The result suggests that there are situations where a higher union inefficiency can increase the firm's profit. For example, a rise in the inefficiency leads to a reduction in the union size. If the elasticity is sufficiently large, this puts sufficient upward pressure on the firm's bargaining power such that the firm's best response is to reduce its costly effort. This in turn can increase the firm's profit. This finding may be important for union and firm wage negotiations if the inefficient coefficient can be influenced.

# Chapter 7

## Costly Firm Bargaining when Wages Increase with Skill

### 7.1 Model Setup

This chapter studies the interaction between a union, firm and workers when output is observable and wages increase with a worker's skill. The model studied in this chapter is similar to the model of Chapter 6; the only difference is that workers earn a wage that increases linearly in their productivity (as in Chapter 4). This model is studied to gain insight into union and firm preferences on performance-based pay and on the observability of output.

As in Chapter 6, the firm's costly bargaining effort is denoted by  $c \geq 0$ , the bargaining function is denoted  $K(c, n)$  and has the properties described in Chapter 6. Assumptions (6.1) through (6.7) hold, and the inefficiency coefficient,  $\varepsilon$ , is exogenously given.

Let  $w(j) = \beta j > 0$  denote the wage for union worker  $j$ . The union bargains with the firm over the rate  $\beta > 0$ . Similarly, non-union workers also earn a wage linearly increasing in their skill in the same fashion as for union workers. The wage rate of non-union workers may be equal to or different from the wage rate of union workers. As explained in the following subsection, in equilibrium these wage rates are the same because of the

incentives of workers.

The Budget Balance constraint of the previous chapters holds here; in particular, wages in aggregate cannot exceed workers' surplus,  $Y_w$ . If all workers receive the wage rate,  $\beta$ , the constraint becomes:

$$Y_w = \sum_{j=1}^{N_0} \beta j = \frac{\beta N_0(N_0 + 1)}{2}.$$

By substituting equation (6.11) for the workers' surplus, this constraint becomes:

$$\frac{\beta N_0(N_0 + 1)}{2} = Y_w = K(c, n)Y = \frac{K(c, n)}{2} [N_0(N_0 + 1) - \varepsilon n(n + 1)]. \quad (7.1)$$

Solving this equation for the wage rate,  $\beta$ , yields:

$$\beta = n\sqrt{2} \frac{\sqrt{N_0(N_0 + 1) - \varepsilon n(n + 1)}}{N_0(N_0 + 1)}. \quad (7.2)$$

This wage rate is similar to the one in Chapter 4 given by equation (4.2); the only difference is that the bargaining function depends on the firm's bargaining effort in addition to the union size.

The problem of the union is to maximize the wage rate,  $\beta$ , by choosing the number of union members,  $n$ , subject to budget balance and the choice of effort that the firm makes. Mathematically:

$$\max_{0 \leq n \leq N_0} \beta \quad \text{subject to condition (7.1).}$$

The problem of the firm is the same as in Chapter 6. In particular, for every level of union membership,  $n$ , the firm chooses the effort,  $c$ , that maximizes its profit, denoted as  $\pi$  and given by equation (6.12). Mathematically, the firm's best reply is given by:

$$c^{BR}(n) = \arg \max_{c \geq 0} \pi(c, n).$$

## 7.2 Nash Equilibrium

As in the other chapters, the focus will be on an equilibrium where all workers earn the same wage rate regardless of their unionization status; that is, the wage rate received by non-union workers is the same as the wage rate of union workers,  $\beta$ . As described in Chapter 4, this is because two different wage rates would generate profitable deviations for workers, and hence, such a strategy profile would not be an equilibrium.

The best replies of the firm and union turn out to be the same as those characterized by Proposition 10 and Proposition 11, respectively, in Chapter 6 (where output was unobservable). These best replies exist and are unique. The best reply of the firm is the same as in Chapter 6 because it does not depend on the wage structure. As for the union's best response, the first order condition of the wage rate with respect to the size of the union,  $n$ , is

$$\frac{\partial K}{\partial n} [N_0(N_0 + 1) - \varepsilon n(n + 1)] = K(c, n)\varepsilon(2n + 1),$$

which is the same first order condition as in Proposition 11. Indeed, the size of the union is robust to the structure of wages.

Because the best response functions for both the firm and union are the same as in Chapter 6, so too is the equilibrium condition of Proposition 12.

The following subsection follows Chapter 6 by studying the game with a restricted bargaining function.

## 7.3 Example: Particular Bargaining Function

Suppose in this example that the bargaining function is a ratio of the union size to the firm's costly effort, as in the example used in Chapter 6 and given by equation (6.17). In particular, the relative bargaining power of workers is the function  $K : [0, +\infty) \times [0, N_0] \rightarrow \mathbb{R}$ , defined at every pair

$(c, n) \in [0, +\infty) \times [0, N_0]$  by:

$$K(c, n) = \frac{n^2}{c + n^2}.$$

The properties of this bargaining function are explained in the Chapter 6 example. The following result establishes the wage rate  $\beta$  as a function of the number of union members, the inefficiency coefficient and the size of the workforce.

**Lemma 8** *Suppose the bargaining function is given by equation (6.17). The wage rate,  $\beta$ , is defined by:*

$$\beta = n\sqrt{2} \frac{\sqrt{N_0(N_0 + 1) - \varepsilon n(n + 1)}}{N_0(N_0 + 1)}.$$

The best reply of the firm is the same as in the Chapter 6 example, given by equation (6.24) of Proposition 13. The best reply of the firm does not depend on the wage structure (see proof of Proposition 13). The next proposition proves the best reply of the union exists and characterizes this best reply.

**Lemma 9** *Suppose the bargaining function is given by equation (6.17). The best reply of the union is the function that takes each  $c > 0$  and returns a unique value of  $n^{BR}(c)$ . Depending on the parameters, the best reply is either  $n^{BR} = N_0$  (all workers are in the union) or the unique interior solution implicitly defined by the first order condition of the union's problem, given by:*

$$n^{BR}[2n^{BR^3} + n^{BR^2} + 4cn^{BR} + 3c] - \frac{2cN_0(N_0 + 1)}{\varepsilon} = 0.$$

The first order condition of Lemma 9 and its unique positive solution  $n^{BR}$  are the same as in Proposition 14 of Chapter 6. Indeed, the best reply of the union  $n^{BR}$  does not depend on whether there is a constant wage or a linearly increasing wage. This is because the solution to the problem is the



same and, hence, the best response is the same. Proposition 14 of Chapter 6 studies the properties of this best reply in more detail.

Because the best replies of the firm and union are the same as in the Chapter 6 example, the comparative static results of that chapter (in particular, those of Proposition 12) also hold here. The only new comparative statics result in this section relates to the equilibrium wage rate,  $\beta^*$ . The following corollary studies this comparative static result.

**Corollary 13** *Suppose the bargaining function is given by equation (6.17) and that bargaining elasticity,  $\gamma_n$ , is constant. In this case, the equilibrium wage rate is increasing in the size of the workforce. Formally:*

$$\frac{d\beta^*}{dN_0} > 0.$$

*The equilibrium wage rate is decreasing in the inefficiency coefficient if the elasticity,  $\gamma_n$ , is sufficiently small. Mathematically:*

$$\frac{d\beta^*}{d\varepsilon} < 0 \quad \Leftrightarrow \quad \gamma_n < \frac{2}{\varepsilon} \left( 1 + \varepsilon - \sqrt{1 + 2\varepsilon} \right).$$

Corollary 13 proves that an increase in the union inefficiency can increase or decrease the equilibrium wage rate depending on the relative magnitudes of the inefficiency coefficient and the elasticity  $\gamma_n$ . This is similar to the result of Corollary 12 which showed the equilibrium wage could increase or decrease with the inefficiency coefficient. The intuition for this result is explained in detail in that corollary. The result of Corollary 13 is implied by Corollary 12, which established when  $d\alpha^*/d\varepsilon < 0$ . This is because  $\beta$  is an increasing transformation of  $\alpha$  (for details, see the proof of Corollary 6).

## 7.4 Summary

This chapter studied the interaction between workers, a union and a firm using a similar model as Chapter 6; the only difference was that output was

observable and wages increased linearly with a worker's productivity. As in Chapter 6, this chapter focused on studying the properties of an equilibrium where lower skilled workers joined the labor union while higher skilled workers did not.

The equilibrium variables under this model were the same as the model of Chapter 6, where output was observable. In other words, the observability of a worker's output did not change the equilibrium variables. We found that the equilibrium wage rate is decreasing in the inefficiency coefficient if the elasticity of the bargaining function with respect to the union size is not too large. The summary section of Chapter 4 explains the implications for the union's preferences on performance pay.

# Chapter 8

## Conclusion

This thesis proposed and analyzed micro-theoretic models of the strategic interaction between heterogeneous workers and a union, and that between a firm and a union. It was assumed that joining the union increased the bargaining power of workers, but affected their productivity. The thesis tried to use the simplest possible methods to capture the phenomenon. It also went beyond a traditional unilateral equilibrium solution concept to check whether coalitional deviations would make sense. The union's preferences on performance-based pay were analyzed by studying the models under different wage structures.

Two main models were analyzed. The first was the Constant Wage Model of Chapter 3, where the strategic players were the union and the workers. The second was the Firm Choice Model of Chapter 6, where the strategic players were the union and the firm. Chapters 4, 5 and 7 studied variations of these baseline models by changing the assumptions on output observability and by incorporating performance-based pay. They also compared the equilibrium variables across the models.

All models assumed a finite number of heterogeneous workers, with each worker deciding whether to join the union. The heterogeneity of workers was in their skill; each worker had a different skill level. Workers and the firm produced a surplus and the union bargained on behalf of workers for

their share of the surplus (which we called their wage). Despite the fact a specific functional form was used for the surplus,  $Y$ , most of the results were qualitative in nature and may potentially work for some generic functions of the surplus that are decreasing in the number of union workers.<sup>1</sup> This is because, in most cases, the proofs of the results were for a generic function of the surplus  $Y(n)$  rather than for the specific functional form. The union determined, and maximized, wages for its members, subject to budget balance; that is, union members shared equally what was left from the workers' surplus after salaries to non-members were paid.

This thesis focused on studying equilibria in which the least productive workers joined the union and the most productive workers did not. Across the models, we showed that this type of equilibrium exists, is unique, and is robust to coalitional deviations. We found expressions for the equilibrium union size, union density, the wage of union and non-union workers, the surplus and profit of the firm and the bargaining power of workers and the firm.

The most striking result that holds across all models is that the firm's surplus and profit can sometimes increase with the inefficiency coefficient of the union (see Corollary 12). This suggests there can exist situations where firms have a perverse incentive to make union workers less efficient compared to non-union workers. The intuition is that the reduced efficiency makes unionized workers want to exit the union and this improves the firm's bargaining power and the total surplus.

We also found that union membership and salaries decrease with the inefficiency coefficient (see Corollary 2, 6 and 12). In other words, unionized workers always prefer to be as efficient as their non-unionized colleagues. We found that union wages were generally lowest when they are linked to a worker's output (see Corollary 10). This suggests that unions may face

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<sup>1</sup>Describing the results as qualitative in nature means that the model is designed to provide intuition and insights on the strategic interactions of the players in the game; it has not been calibrated with data which may be required to obtain results that are more quantitative in nature.

incentives to oppose performance-based pay for their members.

Other key findings included:

- All workers join the labor union if the cost from doing so (in terms of the inefficiency coefficient) is sufficiently small and there is a benefit from joining (in terms of the workers' bargaining power). See Propositions 1 and 7 and Corollary 1. In some cases, the union had to improve the productivity of its members before all workers would unionize (see Proposition 16).
- Some workers, particularly those with relatively low productivity, can remain unionized even when doing so makes them very inefficient. This is because they benefit from the improved bargaining power of the union.
- In the model where the firm chooses its level of effort in bargaining, a higher inefficiency coefficient decreases the equilibrium union size and the firm's bargaining effort (see Proposition 12). The equilibrium union size is decreasing in the inefficiency coefficient because a higher inefficiency effectively increases the cost of unionization. The firm's optimal bargaining effort is decreasing in the inefficiency coefficient because the union size, and hence workers' bargaining power, decrease.
- A higher inefficiency coefficient decreases the equilibrium firm profit if the elasticity of the bargaining function with respect to union size is not too large (see Proposition 12). The result suggests that there are situations where a higher union inefficiency can increase the firm's profit. For example, a rise in the inefficiency leads to a reduction in the union size. If the elasticity is sufficiently large, the size of the union decreases a lot, the workers' bargaining power decreases a lot, and these factors lead to a rise in the surplus and the firm's bargaining power.

## Future Research

Future research may consider how the strategic interaction between unions, workers and firms depends on governments and the legal and economic environment. This may be important because these factors can affect the relative bargaining power of workers and other aspects of the strategic interaction. This thesis allowed for a generic bargaining function that could be expanded to incorporate these factors, especially if it were clear how a government would intervene in bargaining. For example, a pro-union government may implement policies that increase the bargaining power of the union, all else constant. This may lead to a relatively lower union size in equilibrium because the marginal benefit from joining the union is reduced while the cost remains the same. In contrast, an anti-union government may implement policies that decrease the bargaining power of the union, all else constant – this may lead to relatively larger unions (and hence a larger inefficiency) as workers join the union to protect their bargaining power.

Future research may also consider how the strategic interaction depends on the number and types of unions, in light of heterogeneous workers and union inefficiency. This is because some unions represent workers in a particular industry and others may represent workers of a specific firm. Additionally, some firms may bargain with multiple unions. For example, QANTAS Airways (the flag carrier airline of Australia) bargains with multiple unions over salaries. In 2011, workers of three unions (representing engineers, baggage and catering staff, and long-haul pilots) undertook protected industrial action which resulted in QANTAS grounding most of its fleet (ABC [1]). In this framework of multiple unions, timing may become a more relevant factor. For example, unions may want to coordinate industrial action to maximize their bargaining power.

Future research may also investigate how the strategic interaction depends on the production technology. This thesis assumed a technology which made workers perfect substitutes. For instance, if workers were not perfect substitutes, a loss of efficiency for one worker could cause flow-on effects to

other workers. In this case, the impact of any union inefficiencies are magnified; one might therefore expect relatively smaller levels of unionization in this case.

Moreover, this thesis assumed full employment. As described earlier, macroeconomic variables may change the bargaining power of the union. Similarly, recent studies, such as Mitra [39], suggest that international trade could be important for explaining the unionization rate amongst workers. It may therefore be important for future research on labor unions to account for this factor.

# Appendix A

## Proofs

This Appendix contains all proofs. To help reduce the size of some equations in this appendix, let  $x = N_0(N_0 + 1)$ .

### **Proof of Lemma 1**

To see that the function  $\alpha = \alpha(n)$  in equation (3.10) is strictly concave, first express it as:

$$2N_0\alpha = K[x - \varepsilon n(n + 1)].$$

Taking the derivative of this expression with respect to  $n$  yields:

$$\begin{aligned} 2N_0 \frac{d\alpha}{dn} &= K[-\varepsilon(2n + 1)] + [x - \varepsilon n(n + 1)] \frac{dK}{dn} \\ &= \frac{dK}{dn} [x - \varepsilon n(n + 1)] - \varepsilon(2n + 1)K. \end{aligned}$$

Taking the derivative of this expression with respect to  $n$  yields:

$$\begin{aligned} 2N_0 \frac{d^2\alpha}{dn^2} &= \frac{dK}{dn} [-\varepsilon(2n + 1)] + \frac{d^2K}{dn^2} [x - \varepsilon n(n + 1)] - \varepsilon(2n + 1) \frac{dK}{dn} - 2\varepsilon K \\ &= [x - \varepsilon n(n + 1)] \frac{d^2K}{dn^2} - 2\varepsilon(2n + 1) \frac{dK}{dn} - 2\varepsilon K. \end{aligned}$$



Re-arranging for  $d^2\alpha/dn^2$  yields:

$$\frac{d^2\alpha}{dn^2} = \frac{[x - \varepsilon n(n+1)] \frac{d^2K}{dn^2} - 2\varepsilon(2n+1) \frac{dK}{dn} - 2\varepsilon K}{2N_0}.$$

This second derivative is negative because  $d^2K/dn^2 < 0$  (by assumption 3.5),  $dK/dn > 0$  (by assumption 3.4) and  $K \geq 0$  (by assumption 3.1). Hence, the function  $\alpha = \alpha(n)$  in equation (3.10) is strictly concave.

### Proof of Proposition 1

Take the first and second derivatives of  $\alpha$  with respect to  $n$ , in equation (3.10), to obtain:

$$\begin{aligned} \frac{d\alpha}{dn} &= \frac{1}{2} \left( N_0 + 1 - \frac{\varepsilon n(n+1)}{N_0} \right) \frac{dK}{dn} - \frac{(2n+1)\varepsilon K}{2N_0}, \\ \frac{d^2\alpha}{dn^2} &= \frac{1}{2} \left( N_0 + 1 - \frac{\varepsilon n(n+1)}{N_0} \right) \frac{d^2K}{dn^2} - \frac{\varepsilon(2n+1)}{N_0} \frac{dK}{dn} - \frac{\varepsilon K}{N_0} < 0. \end{aligned}$$

This second derivative is always negative because  $N_0 + 1 - \varepsilon n(n+1)/N_0 > 0$ , with strict inequality holding if  $\varepsilon < 1$  or if  $n < N_0$ . Hence,  $d\alpha/dn$  is a decreasing function. Because  $0 \leq K(0) \leq 1$ , then, by assumption (3.3), it turns out that:

$$\frac{d\alpha(0)}{dn} = \frac{N_0 + 1}{2} \frac{dK}{dn}(0) - \frac{\varepsilon K(0)}{2N_0} > 0.$$

Because  $K(0) \geq 0$ ,  $N_0 > 0$ , and  $dK/dn > 0$ , for every  $n$  such that  $0 < n < N_0$ , then  $K(N_0) > 0$ . If function  $\alpha(n)$  is always increasing on its domain, then its unique maximal value is  $n^* = N_0$ . On the other hand, if function  $\alpha(n)$  is decreasing at  $n = N_0$ , then the first order condition establishes the unique maximizer, by the Intermediate Value Theorem (which can be applied because the objective function is continuous). With some algebra, the interior solution  $n^*$  satisfies:

$$(N_0(N_0 + 1) - \varepsilon n^*(n^* + 1)) \frac{dK(n^*)}{dn} = \varepsilon(2n^* + 1)K(n^*).$$

Because  $K(n^*)/n^* > 0$ , and  $N_0(N_0 + 1) > \varepsilon n^*(n^* + 1)$  for every  $n^* < N_0$ , then:

$$\frac{d\alpha(n^*)}{dn} \geq 0 \quad \Leftrightarrow \quad \frac{\frac{dK(n^*)}{dn}}{K(n^*)/n^*} \geq \frac{\varepsilon n^*(2n^* + 1)}{N_0(N_0 + 1) - \varepsilon n^*(n^* + 1)}.$$

At the upper corner  $n^* = N_0$ :

$$\frac{d\alpha(N_0)}{dn} \geq 0 \quad \Leftrightarrow \quad \gamma(N_0) = \frac{\frac{dK(N_0)}{dn}}{K(N_0)/N_0} \geq \frac{\varepsilon N_0(2N_0 + 1)}{(1 - \varepsilon)N_0(N_0 + 1)}. \quad (\text{A.1})$$

Solving for  $\varepsilon$  the inequality on the right-hand side of (A.1) leads to:

$$\varepsilon \leq \frac{\gamma(N_0)N_0(N_0 + 1)}{\gamma(N_0)N_0(N_0 + 1) + N_0(2N_0 + 1)}.$$

As  $x/N_0 = N_0 + 1$ , dividing by  $N_0$  all terms on the top and on the bottom of the fraction on the right-hand side results in equation (3.12).

As discussed in the text, all non-union workers have the correct incentives and the union is maximizing its utility by construction. The only possible deviation is if a union member leaves the union. If this occurred, then  $n^* < N_0$ , so that there is an interior solution for the union's problem. Suppose a worker with skill  $j \leq n^*$  considers leaving the union while all other players are playing according to the prescribed equilibrium. The non-union surplus, denoted  $G_N^j$ , would become:

$$\begin{aligned} G_N^j &= K(n^* - 1) \left( \varepsilon j + (1 - \varepsilon) \sum_{i=1}^{n^*} i + \sum_{i=n^*+1}^{N_0} i \right) - \alpha^*(n^* - 1) \\ &= \frac{K(n^* - 1)}{K(n^*)} Y_w + \varepsilon j K(n^* + 1) - \alpha^*(n^* - 1). \end{aligned}$$

As worker with skill  $j$  moves from inside to outside the union, the workers'

surplus has a net change of  $\varepsilon j K(n^* - 1)$ . By definition (3.8), for  $n = n^*$ ,

$$\begin{aligned}\frac{Y_w}{K(n^*)} &= (1 - \varepsilon) \sum_{i=1}^{n^*} i + \sum_{i=n^*+1}^{N_0} i, \text{ or} \\ \frac{Y_w}{K(n^*)} &= \frac{N_0(N_0 + 1) - \varepsilon n^*(n^* + 1)}{2}.\end{aligned}$$

The non-union workers obtain all the surplus of the workers, except for a wage of  $\alpha^*$  to each union member, for a total payment of  $\alpha^*(n^* - 1)$ . Worker with skill  $j$  has no incentive to deviate if and only if  $G_N^j / (N_0 - n^* + 1) \leq \alpha^*$ ; that is, if:

$$\frac{K(n^* - 1)}{K(n^*)} Y_w + \varepsilon j K(n^* - 1) - \alpha^*(n^* - 1) \leq \alpha^*(N_0 - n^* + 1).$$

Define  $z = x - \varepsilon n^*(n^* + 1)$ . Because  $Y_w / K(n^*) = z/2$ , this inequality is equivalent to:

$$K(n^* - 1) [z + 2\varepsilon j] \leq 2\alpha^* N_0.$$

Substituting  $\alpha^*$  of equation (3.10), with  $n = n^*$ :

$$K(n^* - 1) [z + 2\varepsilon j] \leq 2N_0 \left( \frac{K(n^*)z}{2N_0} \right).$$

This is equivalent to:

$$\frac{K(n^* - 1)}{K(n^*)} \leq \frac{z}{z + 2\varepsilon j}. \quad (\text{A.2})$$

Because (by assumption) the second derivative of function  $K(n)$  is non-positive, function  $n \mapsto dK/dn$  is weakly decreasing, and then,  $dK(n')/dn \leq$

$dK(n^*)/dn$ , for every  $n' \geq n^*$ . By the Fundamental Theorem of Calculus:

$$\begin{aligned}
K(n^* - 1) &= K(n^*) - \int_{n^*-1}^{n^*} \frac{dK(n')}{dn} dn' \\
&\leq K(n^*) - \int_{n^*-1}^{n^*} \frac{dK(n^*)}{dn} dn' \\
&= K(n^*) - \frac{dK(n^*)}{dn} \int_{n^*-1}^{n^*} dn' \\
&= K(n^*) - \frac{dK(n^*)}{dn}.
\end{aligned} \tag{A.3}$$

Dividing both sides by  $K(n^*)$  results in:

$$\frac{K(n^* - 1)}{K(n^*)} \leq 1 - \frac{\frac{dK(n^*)}{dn}}{K(n^*)}. \tag{A.4}$$

Using equation (3.11), this inequality becomes:

$$\begin{aligned}
\frac{K(n^* - 1)}{K(n^*)} &\leq 1 - \frac{\varepsilon(2n^* + 1)}{N_0(N_0 + 1) - \varepsilon n^*(n^* + 1)} \\
\frac{K(n^* - 1)}{K(n^*)} &\leq \frac{z - \varepsilon(2n^* + 1)}{z}.
\end{aligned}$$

Hence, in order to prove that inequality (A.2) always holds, it suffices to prove that:

$$\frac{z - \varepsilon(2n^* + 1)}{z} \leq \frac{z}{z + 2\varepsilon j}.$$

After some algebra, this condition becomes simply:

$$\begin{aligned}
[z - \varepsilon(2n^* + 1)][z + 2\varepsilon j] &\leq z^2 \\
z^2 + 2\varepsilon jz - 2n^*\varepsilon z - 4\varepsilon^2 jn^* - \varepsilon z - 2j\varepsilon^2 &\leq z^2.
\end{aligned}$$

This inequality holds because  $2n^*\varepsilon z > 2\varepsilon jz$ , as  $n^* \geq j$ . This completes the proof.

**Proof of Corollary 1**

By assumption (3.6),  $dK(N_0)/dn = 0$  and, hence,  $\gamma(N_0) = 0$ . If  $\varepsilon = 1$ , then the equivalent conditions in inequality (A.1) must be violated. Suppose  $\varepsilon < 1$ . Because  $\gamma(N_0) = 0$  and  $(1 - \varepsilon)N_0(N_0 + 1) > 0$ , the equivalent conditions in (A.1) become simply  $\varepsilon N_0(2N_0 + 1) \leq 0$ . The only possibility for this to be true is  $\varepsilon = 0$ , but, by assumption,  $\varepsilon > 0$ . Hence, conditions (A.1) are violated, and the objective function is decreasing at  $n = N_0$ . By the Intermediate Value Theorem, there is a unique solution  $n^*$ , which is interior and implicitly defined by equation (3.11).

**Proof of Proposition 2**

Let the coalition be of two or more workers;  $|\zeta| \geq 2$ . This coalition can contain unionized and non-unionized workers, formally:

$$j_U \in \zeta,$$

$$j_N \in \zeta.$$

In particular, there are three possible types of coalitions: (a) a coalition of entirely non-union workers, (b) a coalition of entirely union workers, and (c) a coalition containing both union and non-union workers.

First consider the case where the coalition consists of non-unionized workers only, so that:

$$\zeta \subset \{\text{non-unionized workers}\}.$$

In this case, none of these workers have an incentive to join the union, because they receive  $\alpha^*$  out of the union and  $\alpha^*$  in the union.

Second, consider the case where the coalition consists of unionized workers only. In particular, suppose there are a total of  $c$  union members in the

coalition, where  $c \leq n^*$ , so that:

$$\zeta \subset \{\text{union members}\} = \{n^*, n^* - 1, \dots, n^* - c\}$$

$$|\zeta| = c.$$

If this coalition leaves the union, the surplus accruing to non-union workers, denoted  $G_N^K$ , becomes:

$$G_N^K = K(n^* - c) \frac{Y_w}{K(n^*)} + K(n^* - c) \varepsilon \left( \sum_{i=n^*-c+1}^{n^*} i \right) - \alpha(n^* - c).$$

Because  $Y_w/K(n^*) = z/2$ , this equation is equivalent to:

$$G_N^K = K(n^* - c) \frac{z}{2} + K(n^* - c) \frac{\varepsilon(2n^* - c + 1)c}{2} - \alpha n^* + \alpha c.$$

This coalition has no incentive to leave the union if the following incentive condition holds:

$$G_N^K \leq \alpha N_0 - \alpha n^* + \alpha c.$$

This is equivalent to:

$$K(n^* - c) \frac{z}{2} + K(n^* - c) \frac{\varepsilon(2n^* - c + 1)c}{2} \leq \alpha N_0.$$

Substituting  $\alpha$  of equation (3.10), with  $n = n^*$ :

$$K(n^* - c) [z + \varepsilon(2n^* - c + 1)c] \leq K(n^*)z.$$

With some algebra:

$$\frac{K(n^* - c)}{K(n^*)} \leq \frac{z}{z + \varepsilon(2n^* - c + 1)c}. \quad (\text{A.5})$$

From equation (A.3) in the proof of Proposition 1,

$$\begin{aligned} K(n^*) - K(n^* - c) &\geq c \frac{dK}{dn} \\ 1 - c \frac{dK/dn}{K(n^*)} &\geq \frac{K(n^* - c)}{K(n^*)}. \end{aligned}$$

Substituting equation (3.11) for  $(dK/dn)/K(n^*)$  results in:

$$1 - \frac{c\varepsilon(2n^* + 1)}{z} \geq \frac{K(n^* - c)}{K(n^*)}.$$

Hence, in order to prove that inequality (A.5) always holds, it suffices to prove that:

$$\frac{z - c\varepsilon(2n^* + 1)}{z} \leq \frac{z}{z + \varepsilon(2n^* - c + 1)c}.$$

After some algebra, this becomes:

$$0 \geq -zc\varepsilon(2n^* + 1) + zc\varepsilon(2n^* - c + 1) - \varepsilon^2(2n^* - c + 1)(2n^* + 1)c^2.$$

This inequality holds because  $zc\varepsilon(2n^* + 1) > zc\varepsilon(2n^* - c + 1)$ .

Finally, consider the case where the coalition consists of both unionized and non-unionized workers. In this case, if a non-unionized worker with skill  $j > n^*$  joins the union and a unionized worker with skill  $j \leq n^*$  leaves the union, the workers' bargaining power is unchanged, but the surplus falls and hence wages decline. Because this is true for all union and non-union workers, there is no profitable deviation for a coalition of this form.

### **Proof of Proposition 3**

In this setup, the workers' surplus becomes:

$$\begin{aligned} Y_w &= (1 - \varepsilon)K(N_0 - n^* + 1) \sum_{n^*}^{N_0} j + K(N_0 - n^* + 1) \sum_1^{n^*-1} j \\ &= \frac{K(N_0 - n^* + 1)}{2} [(1 - \varepsilon)N_0(N_0 + 1) + \varepsilon(n^* - 1)n^*]. \end{aligned}$$

The wage is:

$$\alpha = \frac{K(N_0 - n^* + 1)}{2N_0} [(1 - \varepsilon)N_0(N_0 + 1) + \varepsilon(n^* - 1)n^*]. \quad (\text{A.6})$$

Non-union workers have a profitable coalitional deviation with union workers. For example, a non-union worker can switch their membership status with a union worker. In this case, wages rise because the inefficiency is reduced and bargaining power is unchanged. However, non-union workers have no profitable deviation either unilaterally or in a coalition of other non-union workers: they receive  $\alpha$  outside the union and  $\alpha$  inside the union.

As for union workers, they have profitable deviations both unilaterally and in coalitions. To see that a union worker has a unilateral deviation, note that if a union worker leaves the union the workers' surplus has a net change of  $\varepsilon j K(N_0 - n^*)$ . Worker with skill  $j$  has no incentive to deviate if  $G_n/n^* \leq \alpha$ . Substituting  $G_n$  from equation (3.8) yields:

$$\frac{K(N_0 - n^*)Y_w}{K(N_0 - n^* + 1)} + \varepsilon j K(N_0 - n^*) - \alpha(N_0 - n^*) \leq \alpha n^*$$

$$\frac{K(N_0 - n^*)Y_w}{K(N_0 - n^* + 1)} + \varepsilon j K(N_0 - n^*) \leq \alpha N_0.$$

Substituting  $z = [(1 - \varepsilon)N_0(N_0 + 1) + \varepsilon(n^* - 1)n^*]$  yields:

$$K(N_0 - n^*)(z + 2\varepsilon j) \leq 2\alpha N_0.$$

Substituting  $\alpha$  from equation (A.6) yields:

$$K(N_0 - n^*)(z + 2\varepsilon j) \leq K(N_0 - n^* + 1)z.$$

The remainder of the proof follows exactly as in Proposition 1 at equation (A.2). Because each union worker has a profitable unilateral deviation they also have profitable coalitional deviation.

## **Proof of Corollary 2**



*I. Comparative Static Result for the Equilibrium Union Size and the Inefficiency Coefficient*

The equilibrium condition (3.11) may be written as:

$$\frac{n^*}{K}K_n = -\frac{n^*}{Y}Y_n.$$

Equivalently:

$$YK_n = -KY_n.$$

Taking the derivative of both sides of this equality with respect to the inefficiency coefficient yields:

$$YK_{nn}\frac{dn^*}{d\varepsilon} + K_n\left[Y_n\frac{dn^*}{d\varepsilon} + Y_\varepsilon\right] = -Y_nK_n\frac{dn^*}{d\varepsilon} - K\left[Y_{nn}\frac{dn^*}{d\varepsilon} + Y_{n\varepsilon}\right].$$

Isolating  $dn^*/d\varepsilon$  yields:

$$\begin{aligned}\frac{dn^*}{d\varepsilon}[YK_{nn} + 2K_nY_n + Y_{nn}K] &= -KY_{n\varepsilon} - K_nY_\varepsilon \\ \frac{dn^*}{d\varepsilon} &= \frac{-KY_{n\varepsilon} - K_nY_\varepsilon}{YK_{nn} + 2K_nY_n + Y_{nn}K}.\end{aligned}$$

The denominator of the right-hand side is negative because  $K_{nn} < 0$  (by assumption 3.5),  $K_nY_n < 0$  (by assumption 3.4 and because  $Y_n < 0$  by equation 3.7) and  $Y_{nn} < 0$  (by equation 3.7). The numerator is positive because  $KY_{n\varepsilon} > 0$  (by equation 3.7) and because  $K_nY_\varepsilon > 0$  (by assumption 3.4 and equation 3.7). Hence, the derivative  $dn^*/d\varepsilon$  is negative.

In addition (and for use in later proofs), the expression for  $dn^*/d\varepsilon$  for the particular function  $Y$  assumed in this thesis is given by taking the implicit derivative, with respect to  $\varepsilon$ , in both sides of equation (3.11). Undertaking this calculation and isolating  $dn^*/d\varepsilon$ , results in:

$$\frac{dn^*}{d\varepsilon} = \frac{(2n^* + 1)K(n^*) + n^*(n^* + 1)\frac{dK(n^*)}{dn}}{[x - \varepsilon n^*(n^* + 1)]\frac{d^2K(n^*)}{dn^2} - 2\varepsilon(2n^* + 1)\frac{dK(n^*)}{dn} - 2\varepsilon K(n^*)}. \quad (\text{A.7})$$

This expression is negative because the numerator of the fraction on the right-hand side is positive and the denominator is negative, as  $x - \varepsilon n^*(n^* + 1) > 0$ ,  $d^2K(n^*)/dn^2 \leq 0$ ,  $2\varepsilon(2n^* + 1)dK(n^*)/dn > 0$ , and  $2\varepsilon K(n^*) > 0$ .

## II. Comparative Static Result for the Equilibrium Wage and the Inefficiency Coefficient

Because the workers' bargaining power does not depend on the inefficiency coefficient directly, then  $\partial K/\partial \varepsilon = 0$ . Hence:

$$\begin{aligned}\frac{dK(n^*)}{d\varepsilon} &= \frac{\partial K}{\partial \varepsilon} + \frac{dK}{dn} \frac{dn^*}{d\varepsilon} \\ &= \frac{dK}{dn} \frac{dn^*}{d\varepsilon}.\end{aligned}$$

By assumption (3.6), the solution is interior, and equation (3.11) defines the unique  $n^*$ . Taking the derivative, with respect to  $\varepsilon$ , in both sides of equation (3.10), results in:

$$\begin{aligned}2N_0 \frac{d\alpha^*}{d\varepsilon} &= [x - \varepsilon n^*(n^* + 1)] \frac{dK(n^*)}{d\varepsilon} - K(n^*) \left( \varepsilon(2n^* + 1) \frac{dn^*}{d\varepsilon} + n^*(n^* + 1) \right) \\ &= [x - \varepsilon n^*(n^* + 1)] \frac{dK}{dn} \frac{dn^*}{d\varepsilon} - K(n^*) \left( \varepsilon(2n^* + 1) \frac{dn^*}{d\varepsilon} + n^*(n^* + 1) \right).\end{aligned}$$

Thus,  $d\alpha^*/d\varepsilon < 0$  if and only if:

$$[x - \varepsilon n^*(n^* + 1)] \frac{dn^*}{d\varepsilon} \frac{\frac{dK}{dn}}{K(n^*)} < \varepsilon(2n^* + 1) \frac{dn^*}{d\varepsilon} + n^*(n^* + 1).$$

By equation (3.11), this inequality is equivalent to:

$$[x - \varepsilon n^*(n^* + 1)] \frac{dn^*}{d\varepsilon} \frac{\varepsilon(2n^* + 1)}{x - \varepsilon n^*(n^* + 1)} < \varepsilon(2n^* + 1) \frac{dn^*}{d\varepsilon} + n^*(n^* + 1).$$

Cancelling the positive factor  $x - \varepsilon n^*(n^* + 1)$  on the left-hand side yields:

$$\varepsilon(2n^* + 1) \frac{dn^*}{d\varepsilon} < \varepsilon(2n^* + 1) \frac{dn^*}{d\varepsilon} + n^*(n^* + 1).$$

This inequality always holds because  $n^*(n^* + 1) > 0$ . Therefore,  $d\alpha^*/d\varepsilon < 0$ .

### III. Comparative Static Result for Equilibrium Workers' Surplus and Inefficiency Coefficient

Because  $N_0$  is a positive constant,  $Y_w^* = N_0\alpha^*$  and  $d\alpha^*/d\varepsilon < 0$ , then  $dY_w^*/d\varepsilon < 0$ .

### IV. Comparative Static Result for Equilibrium Total Surplus and Inefficiency Coefficient

Taking the derivative of the total surplus, given by equation (3.7), with respect to the inefficiency coefficient yields:

$$\frac{dY^*}{d\varepsilon} = -\frac{\varepsilon(2n^* + 1)\frac{dn^*}{d\varepsilon} + n^*(n^* + 1)}{2}.$$

Substituting equation (A.7) for  $dn^*/d\varepsilon$  into this equation yields:

$$\begin{aligned} \frac{dY^*}{d\varepsilon} = & -\frac{1}{2}\varepsilon(2n^* + 1) \left[ \frac{(2n^* + 1)K(n^*) + n^*(n^* + 1)\frac{dK(n^*)}{dn}}{[x - \varepsilon n^*(n^* + 1)]\frac{d^2K(n^*)}{dn^2} - 2\varepsilon(2n^* + 1)\frac{dK(n^*)}{dn} - 2\varepsilon K(n^*)} \right] \\ & - \frac{1}{2}n^*(n^* + 1). \end{aligned}$$

The derivative is negative if the term on the right-hand side of the equality is negative. This condition may be written as:

$$\begin{aligned} -\frac{1}{2}\varepsilon(2n^* + 1) \left[ \frac{(2n^* + 1)K(n^*) + n^*(n^* + 1)\frac{dK(n^*)}{dn}}{[x - \varepsilon n^*(n^* + 1)]\frac{d^2K(n^*)}{dn^2} - 2\varepsilon(2n^* + 1)\frac{dK(n^*)}{dn} - 2\varepsilon K(n^*)} \right] \\ - \frac{1}{2}n^*(n^* + 1) < 0. \end{aligned}$$

Multiplying the inequality by the denominator of  $dn^*/d\varepsilon$  (the denominator of  $dn^*/d\varepsilon$  is negative, as described earlier in the proof), this condition may

be written as:

$$\begin{aligned}
& -\frac{1}{2}\varepsilon(2n^*+1)\left[(2n^*+1)K(n^*)+n^*(n^*+1)\frac{dK(n^*)}{dn}\right] \\
& -\frac{1}{2}n^*(n^*+1)\left[\left[x-\varepsilon n^*(n^*+1)\right]\frac{d^2K(n^*)}{dn^2}-2\varepsilon(2n^*+1)\frac{dK(n^*)}{dn}-2\varepsilon K(n^*)\right] \\
& > 0.
\end{aligned}$$

Collecting the  $dK(n^*)/dn$  terms yields:

$$\begin{aligned}
& \frac{dK(n^*)}{dn}[-n^*(n^*+1)\varepsilon(2n^*+1)+n^*(n^*+1)2\varepsilon(2n^*+1)] \\
& -\varepsilon(2n^*+1)^2K(n^*)-n^*(n^*+1)\left[\left[x-\varepsilon n^*(n^*+1)\right]\frac{d^2K(n^*)}{dn^2}-2\varepsilon K(n^*)\right] > 0.
\end{aligned}$$

Dividing the inequality by  $\varepsilon(2n^*+1)$  yields:

$$\begin{aligned}
& \frac{dK(n^*)}{dn}[n^*(n^*+1)]+K(n^*)[-(2n^*+1)+n^*] \\
& -\frac{n^*(n^*+1)[x-\varepsilon n^*(n^*+1)]}{\varepsilon(2n^*+1)}\frac{d^2K(n^*)}{dn^2} > 0.
\end{aligned}$$

Equivalently:

$$\begin{aligned}
& \frac{dK(n^*)}{dn}n^*(n^*+1) > \frac{n^*(n^*+1)[x-\varepsilon n^*(n^*+1)]}{\varepsilon(2n^*+1)}\frac{d^2K(n^*)}{dn^2} \\
& +K(n^*)[(2n^*+1)-n^*].
\end{aligned}$$

Dividing both sides by  $n^*(n^*+1)$  yields:

$$\frac{dK(n^*)}{dn} > \frac{d^2K(n^*)}{dn^2}\frac{x-\varepsilon n^*(n^*+1)}{\varepsilon(2n^*+1)}+K(n^*)\frac{1}{n^*}.$$

After some algebra, this inequality becomes:

$$\begin{aligned} \frac{n^*}{K(n^*)} \frac{dK(n^*)}{dn} &> \frac{\frac{dK(n^*)}{dn}}{\frac{dK(n^*)}{dn}} \frac{n^*}{K(n^*)} \frac{d^2K(n^*)}{dn^2} \frac{x - \varepsilon n^*(n^* + 1)}{\varepsilon(2n^* + 1)} + 1 \\ \gamma(n^*) &> 1 + E_{K,n} \frac{dK(n^*)}{dn} \frac{x - \varepsilon n^*(n^* + 1)}{\varepsilon(2n^* + 1)K(n^*)}. \end{aligned}$$

*V. Comparative Static Result for Equilibrium Firm Surplus and Inefficiency Coefficient*

Taking the derivative of the firm surplus, given by  $Y_f^* = (1 - K(n^*))Y^*$ , with respect to the inefficiency yields:

$$\begin{aligned} \frac{dY_f^*}{d\varepsilon} &= \frac{d(1 - K(n^*))}{d\varepsilon} \frac{x - \varepsilon n^*(n^* + 1)}{2} \\ &\quad - \frac{1 - K(n^*)}{2} \left[ \varepsilon(2n^* + 1) \frac{dn^*}{d\varepsilon} + n^*(n^* + 1) \right]. \end{aligned}$$

This derivative is negative if the right-hand side of the equality is negative.

This condition may be written as:

$$\frac{d(1 - K(n^*))}{d\varepsilon} \frac{x - \varepsilon n^*(n^* + 1)}{1 - K(n^*)} - n^*(n^* + 1) - \varepsilon(2n^* + 1) \frac{dn^*}{d\varepsilon} < 0.$$

Using  $dK/d\varepsilon = (dK/dn)(dn/d\varepsilon)$  this inequality becomes:

$$\begin{aligned} -\frac{dK(n^*)}{dn} \frac{dn^*}{d\varepsilon} \frac{x - \varepsilon n^*(n^* + 1)}{1 - K(n^*)} - n^*(n^* + 1) - \varepsilon(2n^* + 1) \frac{dn^*}{d\varepsilon} &< 0 \\ -\frac{dn^*}{d\varepsilon} \left[ \frac{dK(n^*)}{dn} \frac{x - \varepsilon n^*(n^* + 1)}{1 - K(n^*)} + \varepsilon(2n^* + 1) \right] - n^*(n^* + 1) &< 0. \end{aligned}$$

As  $N_0$  grows large this becomes:

$$-\left[ \frac{x - \varepsilon n^{*2}}{1 - K(n^*)} \lim_{N_0 \rightarrow +\infty} \frac{dK(n^*)}{dn} + \varepsilon 2n^* \right] \lim_{N_0 \rightarrow +\infty} \frac{dn^*}{d\varepsilon} - n^{*2} < 0.$$

Taking the limit of  $dn^*/d\varepsilon$ , given by equation (A.7), as  $N_0$  grows large and

substituting it into this inequality yields:

$$- \frac{\left[ 2n^* K(n^*) + n^{*2} \lim_{N_0 \rightarrow +\infty} \frac{dK(n^*)}{dn} \right] \left[ \frac{x - \varepsilon n^{*2}}{1 - K(n^*)} \lim_{N_0 \rightarrow +\infty} \frac{dK(n^*)}{dn} + \varepsilon 2n^* \right]}{(x - \varepsilon n^{*2}) \lim_{N_0 \rightarrow +\infty} \frac{d^2 K(n^*)}{dn^2} - 4\varepsilon n^* \lim_{N_0 \rightarrow +\infty} \frac{dK(n^*)}{dn}} - n^{*2} < 0.$$

Multiplying the inequality by the denominator of  $\lim_{N_0 \rightarrow +\infty} dn^*/d\varepsilon$  (this denominator is negative, as described earlier in the proof), this condition may be written as:

$$- \left[ 2n^* K(n^*) + n^{*2} \lim_{N_0 \rightarrow +\infty} \frac{dK(n^*)}{dn} \right] \left[ \frac{x - \varepsilon n^{*2}}{1 - K(n^*)} \lim_{N_0 \rightarrow +\infty} \frac{dK(n^*)}{dn} + 2\varepsilon n^* \right] - n^{*2} \left[ (x - \varepsilon n^{*2}) \lim_{N_0 \rightarrow +\infty} \frac{d^2 K(n^*)}{dn^2} - 4\varepsilon n^* \lim_{N_0 \rightarrow +\infty} \frac{dK(n^*)}{dn} \right] > 0.$$

Expanding the terms on the left-hand side of the inequality yields:

$$\begin{aligned} & - \frac{2n^* K(n^*) (x - \varepsilon n^{*2})}{1 - K(n^*)} \lim_{N_0 \rightarrow +\infty} \frac{dK(n^*)}{dn} - 4\varepsilon n^{*2} K(n^*) \\ & - n^{*2} \left( \lim_{N_0 \rightarrow +\infty} \frac{dK(n^*)}{dn} \right)^2 \frac{x - \varepsilon n^{*2}}{1 - K(n^*)} - 2\varepsilon n^{*3} \lim_{N_0 \rightarrow +\infty} \frac{dK(n^*)}{dn} \\ & - n^{*2} (x - \varepsilon n^{*2}) \lim_{N_0 \rightarrow +\infty} \frac{d^2 K(n^*)}{dn^2} + 4\varepsilon n^{*3} \lim_{N_0 \rightarrow +\infty} \frac{dK(n^*)}{dn} > 0. \end{aligned}$$

Collecting like terms yields:

$$\begin{aligned} & - \left( \frac{2K(n^*) (x - \varepsilon n^{*2})}{1 - K(n^*)} - 2\varepsilon n^{*2} \right) \lim_{N_0 \rightarrow +\infty} \frac{dK(n^*)}{dn} - 4\varepsilon n^* K(n^*) \\ & - n^* \left( \lim_{N_0 \rightarrow +\infty} \frac{dK(n^*)}{dn} \right)^2 \frac{x - \varepsilon n^{*2}}{1 - K(n^*)} > n^* (x - \varepsilon n^{*2}) \lim_{N_0 \rightarrow +\infty} \frac{d^2 K(n^*)}{dn^2}. \end{aligned}$$

Dividing by  $\lim_{N_0 \rightarrow +\infty} (dK(n^*)/dn) (x - \varepsilon n^{*2})$ , which allows the inequality

to be expressed in terms of elasticities, yields:

$$\begin{aligned}
& -\frac{2K(n^*)}{1-K(n^*)} + \frac{2\varepsilon n^{*2}}{x-\varepsilon n^{*2}} - \frac{4\varepsilon n^* K(n^*)}{(x-\varepsilon n^{*2}) \lim_{N_0 \rightarrow +\infty} dK(n^*)/dn} \\
& + \frac{n^*}{1-K(n^*)} \lim_{N_0 \rightarrow +\infty} \frac{d(1-K(n^*))}{dn} > n^* \frac{\lim_{N_0 \rightarrow +\infty} d^2 K(n^*)/dn^2}{\lim_{N_0 \rightarrow +\infty} dK(n^*)/dn}.
\end{aligned}$$

Moving some terms on the left-hand side of the inequality to the right-hand side yields:

$$\begin{aligned}
& \frac{n^*}{1-K(n^*)} \lim_{N_0 \rightarrow +\infty} \frac{d(1-K(n^*))}{dn} > n^* \frac{\lim_{N_0 \rightarrow +\infty} d^2 K(n^*)/dn^2}{\lim_{N_0 \rightarrow +\infty} dK(n^*)/dn} \\
& + \frac{2K(n^*)}{1-K(n^*)} - \frac{2\varepsilon n^{*2}}{x-\varepsilon n^{*2}} + \frac{4\varepsilon n^* K(n^*)}{(x-\varepsilon n^{*2}) \lim_{N_0 \rightarrow +\infty} dK(n^*)/dn}.
\end{aligned}$$

Hence:

$$-\gamma(n) > E_{K_{n,n}} + \frac{2K(n^*)}{1-K(n^*)} - \frac{2\varepsilon n^{*2}}{x-\varepsilon n^{*2}} + \frac{4\varepsilon n^* K(n^*)}{(x-\varepsilon n^{*2}) \lim_{N_0 \rightarrow +\infty} dK(n^*)/dn}.$$

Dividing the numerator and denominator of the last term by  $N_0^2$  yields:

$$-\gamma(n) > E_{K_{n,n}} + \frac{2K(n^*)}{1-K(n^*)} - \frac{2\varepsilon n^{*2}}{x-\varepsilon n^{*2}} + \frac{4\varepsilon \frac{n^*}{N_0^2} K(n^*)}{\left(\frac{x-\varepsilon n^{*2}}{N_0^2}\right) \lim_{N_0 \rightarrow +\infty} dK(n^*)/dn}.$$

Because  $N_0$  is large, this inequality may be written as:

$$-\gamma(n) > E_{K_{n,n}} + \frac{2K(n^*)}{1-K(n^*)} - \frac{2\varepsilon n^{*2}}{N_0^2 - \varepsilon n^{*2}}.$$

#### **Proof of Proposition 4**

Substituting  $\gamma = n(dK/dn)/K$  into the equilibrium equation (3.11) leads

to the polynomial equation:

$$\varepsilon(\gamma + 2)n^2 + \varepsilon(\gamma + 1)n - \gamma N_0(N_0 + 1) = 0.$$

The only positive solution to this polynomial equation is given by equation (3.14).

**Proof of Corollary 3**

Substitute  $n = N_0$  into the equation  $\varepsilon(\gamma + 2)n^2 + \varepsilon(\gamma + 1)n - \gamma N_0(N_0 + 1) = 0$  and cancel a common factor  $N_0$  to find:

$$\varepsilon(\gamma + 2)N_0 + \varepsilon(\gamma + 1) = \gamma(N_0 + 1).$$

Solving this equation for  $\varepsilon$  leads to:

$$\begin{aligned} \varepsilon &= \frac{\gamma(N_0 + 1)}{N_0(\gamma + 2) + (\gamma + 1)} \\ &= \frac{\gamma(N_0 + 1)}{(\gamma + 1) \left[ N_0 \left( \frac{\gamma + 2}{\gamma + 1} \right) + 1 \right]}. \end{aligned}$$

For inefficiency coefficients above this point the union prefers less than full membership.

**Proof of Corollary 4**

By Corollary 2 the workers' surplus is always declining in the inefficiency. Hence, the equilibrium firm surplus is rising in the inefficiency if the equilibrium total surplus is unchanged in the inefficiency. By Corollary 2, the equilibrium total surplus is unchanged in the inefficiency if the following condition holds:

$$\gamma(n^*) = 1 + E_{K,n} \frac{dK(n^*)}{dn} \frac{x - \varepsilon n^*(n^* + 1)}{\varepsilon(2n^* + 1)K(n^*)}.$$



Using the definitions of the elasticities  $\gamma(n^*)$  and  $E_{K,n}$  yields:

$$\frac{n^*}{K(n^*)} \frac{dK(n^*)}{dn} = \frac{\frac{dK(n^*)}{dn}}{\frac{dK(n^*)}{dn}} \frac{n^*}{K(n^*)} \frac{d^2 K(n^*)}{dn^2} \frac{x - \varepsilon n^*(n^* + 1)}{\varepsilon(2n^* + 1)} + 1.$$

Equivalently:

$$\gamma(n^*) = \frac{\gamma(n^*)}{\frac{dK(n^*)}{dn}} \frac{d^2 K(n^*)}{dn^2} \frac{x - \varepsilon n^*(n^* + 1)}{\varepsilon(2n^* + 1)} + 1.$$

Using  $dK(n^*)/dn = \gamma n^{*\gamma-1}/N_0^\gamma$  and  $d^2 K(n^*)/dn^2 = \gamma(\gamma-1)n^{*\gamma-2}/N_0^\gamma$  this condition becomes:

$$\gamma = \frac{\gamma N_0^\gamma}{\gamma n^{*\gamma-1}} \frac{\gamma(\gamma-1)n^{*\gamma-2}}{N_0^\gamma} \frac{x - \varepsilon n^*(n^* + 1)}{\varepsilon(2n^* + 1)} + 1.$$

Thus:

$$\gamma - 1 = \frac{\gamma(\gamma-1)}{n^*} \frac{x - \varepsilon n^*(n^* + 1)}{\varepsilon(2n^* + 1)}.$$

Hence:

$$1 = \frac{\gamma}{n^*} \frac{x - \varepsilon n^*(n^* + 1)}{\varepsilon(2n^* + 1)}.$$

With some algebra:

$$n^* \varepsilon(2n^* + 1) = \gamma (x - \varepsilon n^*(n^* + 1)).$$

Let  $\eta$  denote union density, that is  $\eta = n/N_0$ . Dividing the above equality by  $N_0^2$  and taking the limit as  $N_0$  grows large yields:

$$\varepsilon \eta^{*2} \left( \frac{2}{\gamma} + 1 \right) = 1. \quad (\text{A.8})$$

To obtain  $\eta^*$  (so it can be substituted into the above equality) note that

from equation (3.14) the equilibrium union size is:

$$n^* = \frac{\gamma + 1}{2\gamma + 4} \left( -1 + \sqrt{1 + \frac{4N_0(N_0 + 1)\gamma(\gamma + 2)}{\varepsilon(\gamma + 1)^2}} \right).$$

Dividing by  $N_0$  yields:

$$\eta^* = \frac{\gamma + 1}{N_0(2\gamma + 4)} \left( -1 + \sqrt{1 + \frac{4N_0(N_0 + 1)\gamma(\gamma + 2)}{\varepsilon(\gamma + 1)^2}} \right).$$

As  $N_0$  grows large this becomes:

$$\eta^* = \frac{\gamma + 1}{N_0(2\gamma + 4)} \sqrt{1 + \frac{4N_0(N_0 + 1)\gamma(\gamma + 2)}{\varepsilon(\gamma + 1)^2}}.$$

Substituting this value for  $\eta^*$  into equation (A.8) yields:

$$\varepsilon \left( \frac{\gamma + 1}{N_0(2\gamma + 4)} \right)^2 \left( 1 + \frac{4N_0(N_0 + 1)\gamma(\gamma + 2)}{\varepsilon(\gamma + 1)^2} \right) \left( \frac{2}{\gamma} + 1 \right) = 1.$$

After some algebra and because  $N_0$  grows large this can be written as:

$$\left( \frac{1}{2\gamma + 4} \right)^2 \left( \frac{4\gamma(\gamma + 2)}{1} \right) \left( \frac{2}{\gamma} + 1 \right) = 1.$$

Then:

$$4(\gamma + 2)^2 \left( \frac{1}{2\gamma + 4} \right)^2 = 1.$$

This can be written as:

$$\frac{1}{4\gamma^2 + 16\gamma + 16} = \frac{1}{4(\gamma^2 + 4\gamma + 4)}.$$

Because this equality is satisfied the proof is complete; the equilibrium total surplus is unchanged in the inefficiency coefficient.

### **Proof of Proposition 5**

Substituting  $K(n) = cn$  and  $dK(n)/dn = c$  into equation (3.11) and solving for  $n^*$  results in:

$$n^* = -\frac{1}{3} + \sqrt{\frac{1}{9} + \frac{x}{3\varepsilon}}. \quad (\text{A.9})$$

To get the full unionization condition, let  $n^* = N_0$  in the above equation and solve for  $\varepsilon$ , yielding:

$$\varepsilon > \frac{1}{3} \frac{x}{x - \frac{1}{3}N_0}.$$

To see that  $\varepsilon > 1/3$  implies  $n^* < N_0$  when the workforce size grows large, note that by equation (A.9) the inequality  $n^* < N_0$  is equivalent to:

$$-\frac{1}{3} + \sqrt{\frac{1}{9} + \frac{x}{3\varepsilon}} < N_0.$$

After some algebra, this last inequality becomes  $\varepsilon > 1/3$ . Consequently, if  $\varepsilon \leq 1/3$ , then all workers are unionized. The equilibrium union density is given by dividing both sides of equation (A.9) by  $N_0$ , resulting in:

$$\frac{n^*}{N_0} = -\frac{1}{3N_0} + \sqrt{\frac{1}{9N_0^2} + \frac{3(N_0 + 1)}{\varepsilon N_0}}.$$

Taking the limit as  $N_0 \rightarrow +\infty$  proves the result. As discussed in the text, all non-members have the correct incentives and the union is maximizing its utility by construction. The only possible deviation is if a union member leaves the union. If this occurs, then  $n^* < N_0$ , so that there is an interior solution for the union's problem. Suppose a worker with skill  $j < n^*$  considers leaving the union while all other players are playing according to the prescribed equilibrium. In this case, the proof is the same as in Proposition 1, however the analogous of equation (A.4) will hold with equality here.

### **Proof of Corollary 5**

*I. Comparative Static Result for the Equilibrium Union Size and Inefficiency Coefficient*

To show  $dn^*/d\varepsilon < 0$ , note that the expression for  $n^*$  in Proposition 5 is clearly decreasing in the inefficiency.

## *II. Comparative Static Result for the Equilibrium Wage and Inefficiency Coefficient*

The proof for  $d\alpha^*/d\varepsilon < 0$  is the same as in Corollary 2.

## *III. Comparative Static Result for the Equilibrium Union Size and Workforce Size*

To show  $dn^*/dN_0 > 0$  note that take the expression for  $n^*$  in Proposition 5 is clearly increasing in the workforce size.

## *IV. Comparative Static Result for the Equilibrium Wage and Workforce Size*

To show  $d\alpha^*/dN_0 > 0$ , note that because the union density remains unchanged as  $N_0$  rises, so too does the bargaining power. As such, when the workforce size rises, the share of the surplus going to workers is unchanged, but the total surplus rises. Therefore, the wage must rise.

## *V. Comparative Static Result for the Equilibrium Union Density and Workforce Size*

Note that dividing equation (3.11) by  $N_0$  and taking the derivative with respect to  $N_0$  results in:

$$\frac{d(n^*/N_0)}{dN_0} = \frac{-\sqrt{\frac{1}{9} + \frac{x}{3\varepsilon}} + \frac{N_0(2N_0+1)}{6\varepsilon\sqrt{\frac{1}{9} + \frac{x}{3\varepsilon}}} + \frac{1}{3}}{N_0^2}.$$

This derivative is equal to zero if:

$$\sqrt{\frac{1}{9} + \frac{x}{3\varepsilon}} = \frac{N_0(2N_0+1)}{6\varepsilon\sqrt{\frac{1}{9} + \frac{x}{3\varepsilon}}} + \frac{1}{3}.$$

Equivalently:

$$\frac{1}{9} + \frac{x}{3\varepsilon} = \frac{N_0(2N_0+1)}{6\varepsilon} + \frac{1}{3}\sqrt{\frac{1}{9} + \frac{x}{3\varepsilon}}.$$

Dividing each side by  $N_0^2$  and taking the limit of this equality as  $N_0$  grows

large yields:

$$\frac{1}{3\varepsilon} = \frac{1}{3\varepsilon}.$$

Because the equality is satisfied the derivative is equal to zero and the equilibrium union density does not change with the workforce size.

#### *VI. Comparative Static Result for the Equilibrium Total Surplus and the Inefficiency Coefficient*

Note that from the proof of Corollary 2:

$$\lim_{N_0 \rightarrow +\infty} \frac{dY^*}{d\varepsilon} = -n^{*2} - \left[ \lim_{N_0 \rightarrow +\infty} \frac{dn^*}{d\varepsilon} \right] \varepsilon 2n^*.$$

Substituting  $K(n^*) = cn^*$  and  $dK(n^*)/dn = c$  into the equation for  $dn^*/d\varepsilon$  in the proof of Corollary 2 and taking the limit as  $N_0$  grows large yields:

$$\lim_{N_0 \rightarrow +\infty} \frac{dn^*}{d\varepsilon} = \frac{2n^{*2}c + n^{*2}c}{-4n^*\varepsilon - 2n^*\varepsilon c} = \frac{3n^{*2}c}{-6n^*\varepsilon} = \frac{-n^*}{2\varepsilon}.$$

Substituting this into  $\lim_{N_0 \rightarrow +\infty} dY^*/d\varepsilon$  yields:

$$\lim_{N_0 \rightarrow +\infty} \frac{dY^*}{d\varepsilon} = -n^{*2} - \frac{-n^*}{2\varepsilon} \varepsilon 2n^* = 0.$$

Because this derivative is equal to zero the equilibrium total surplus (when the workforce size is large) does not change with the inefficiency coefficient.

#### **Proof of Lemma 2**

Combining equation (3.8) for the workers' surplus with equation (4.1) for the Budget Balance constraint yields:

$$\beta \sum_{j=1}^{N_0} j = \frac{K(n)}{2} [N_0(N_0 + 1) - \varepsilon n(n + 1)].$$

Solving this equation for the wage rate,  $\beta$ , yields:

$$\beta = K(n) \left( 1 - \varepsilon \frac{n(n + 1)}{N_0(N_0 + 1)} \right).$$

## Proof of Proposition 6

### I. Solution to the Union's Problem

The number of union members is defined by:

$$\frac{d\beta(n^*)}{dn} = 0 \Leftrightarrow \frac{dK(n^*)}{dn} \left(1 - \varepsilon \frac{n^*(n^* + 1)}{N_0(N_0 + 1)}\right) - K(n^*)\varepsilon \frac{2n^* + 1}{N_0(N_0 + 1)} = 0.$$

Then:

$$\frac{dK(n^*)}{dn} \frac{n^*}{K(n^*)} = \frac{\varepsilon n^* (2n^* + 1)}{[N_0(N_0 + 1) - \varepsilon n^*(n^* + 1)]}.$$

The next step is to check that this  $n^*$  is a maximum of the union's problem.

The first and second derivatives are:

$$\frac{d\beta^*}{dn} = \frac{dK}{dn} \left(1 - \varepsilon \frac{n^*(n^* + 1)}{N_0(N_0 + 1)}\right) - K\varepsilon \frac{2n^* + 1}{N_0(N_0 + 1)},$$

$$\frac{d^2\beta^*}{dn^2} = \frac{d^2K}{dn^2} \left(1 - \varepsilon \frac{n^*(n^* + 1)}{N_0(N_0 + 1)}\right) - 2\frac{dK}{dn} \left(\varepsilon \frac{2n^* + 1}{N_0(N_0 + 1)}\right) - \frac{2K\varepsilon}{N_0(N_0 + 1)}.$$

The second derivative is always negative because  $1 - \varepsilon n^*(n^* + 1)/N_0(N_0 + 1) > 0$ . Hence  $d\beta^*/dn$  is a decreasing function. Because  $0 \leq K(0) \leq 1$ , then, by assumption (3.3) it turns out that:

$$\frac{d\beta(0)}{dn} = \frac{dK(0)}{dn} \left(1 - \varepsilon \frac{n^*(n^* + 1)}{N_0(N_0 + 1)}\right) - K(0)\varepsilon \frac{2n^* + 1}{N_0(N_0 + 1)} > 0.$$

Because  $K(0) \geq 0$ ,  $N_0 > 0$ , and  $dK/dn > 0$ , for every  $n^*$  such that  $0 < n^* < N_0$ , then  $K(N_0) > 0$ . If function  $\beta(n^*)$  is always increasing on its domain, then its unique maximal value is  $n^* = N_0$ . On the other hand, if function  $\beta(n^*)$  is decreasing at  $n^* = N_0$ , then the first order condition establishes the unique maximizer, by the Intermediate Value Theorem (which can be applied because the objective function is continuous). With some algebra, the interior solution  $n^*$  satisfies:

$$[x - \varepsilon n^*(n^* + 1)] \frac{dK(n^*)}{dn} = \varepsilon (2n^* + 1) K(n^*).$$

Because  $K(n^*)/n^* > 0$ , and  $x > \varepsilon n^*(n^* + 1)$  for every  $n^* < N_0$ , then:

$$\frac{d\beta(n^*)}{dn} \geq 0 \quad \Leftrightarrow \quad \frac{dK(n^*)}{dn} \frac{n^*}{K(n^*)} \geq \frac{\varepsilon n^*(2n^* + 1)}{x - \varepsilon n^*(n^* + 1)}. \quad (\text{A.10})$$

At the upper corner  $n^* = N_0$ :

$$\frac{d\beta(N_0)}{dn} \geq 0 \quad \Leftrightarrow \quad \frac{dK(N_0)}{dn} \frac{N_0}{K(N_0)} \geq \frac{\varepsilon N_0(2N_0 + 1)}{(1 - \varepsilon)x}.$$

Solving for  $\varepsilon$  the inequality on the right-hand side leads to:

$$\varepsilon \leq \frac{\gamma(N_0)x}{\gamma(N_0)x + N_0(2N_0 + 1)}.$$

Because  $x/N_0 = N_0 + 1$ , dividing by  $N_0$  all terms on the denominator and numerator on the right-hand side results in:

$$\varepsilon \leq \frac{(N_0 + 1)\gamma(N_0)}{(N_0 + 1)\gamma(N_0) + 2N_0 + 1}.$$

## II. Individual Deviations

The next step is to check the incentives of workers. For a non-union member earning  $\beta(n^*)j$ , her outside option is to join the union and also earn  $\beta(n^*)j$ , so she has no profitable deviation. The only non-trivial incentive to check is that of union workers,  $j \leq n^*$ . We want to show that her earnings inside the union  $\beta(n^*)j$  are greater than or equal to her earnings if she left the union,  $\beta(n^* - 1)j$ . Mathematically we want to show that:

$$K(n^*) \left( 1 - \varepsilon \frac{n^*(n^* + 1)}{N_0(N_0 + 1)} \right) j \geq K(n^* - 1) \left( 1 - \varepsilon \frac{n^*(n^* - 1)}{N_0(N_0 + 1)} \right) j$$

$$\frac{x - \varepsilon n^*(n^* + 1)}{x - \varepsilon n^*(n^* - 1)} \geq \frac{K(n^* - 1)}{K(n^*)}.$$

Define  $z = x - \varepsilon n(n + 1)$ , then:

$$\frac{z}{z + 2\varepsilon n^*} \geq \frac{K(n^* - 1)}{K(n^*)}. \quad (\text{A.11})$$

Because (by assumption 3.5) the second derivative of function  $K(n)$  is non-positive, function  $n \mapsto dK/dn$  is weakly decreasing, and then,  $dK(n')/dn \leq dK(n^*)/dn$ , for every  $n' \geq n^*$ . By the Fundamental Theorem of Calculus:

$$\begin{aligned} K(n^* - 1) &= K(n^*) - \int_{n^*-1}^{n^*} \frac{dK(n')}{dn} dn' \\ &\leq K(n^*) - \int_{n^*-1}^{n^*} \frac{dK(n^*)}{dn} dn' \\ &= K(n^*) - \frac{dK(n^*)}{dn} \int_{n^*-1}^{n^*} dn' \\ &= K(n^*) - \frac{dK(n^*)}{dn}. \end{aligned}$$

Dividing both sides by  $K(n^*)$  results in:

$$\frac{K(n^* - 1)}{K(n^*)} \leq 1 - \frac{\frac{dK(n^*)}{dn}}{K(n^*)}.$$

Using equation (A.10), this inequality becomes:

$$\frac{K(n^* - 1)}{K(n^*)} \leq 1 - \frac{\varepsilon(2n^* + 1)}{x - \varepsilon n^*(n^* + 1)}$$

$$\frac{K(n^* - 1)}{K(n^*)} \leq \frac{z - \varepsilon(2n^* + 1)}{z}$$

Hence, in order to prove that inequality (A.11) always holds, it suffices to



prove that:

$$\frac{z - \varepsilon(2n^* + 1)}{z} \leq \frac{z}{z + 2\varepsilon n^*}$$

$$-4\varepsilon^2 n^{*2} - \varepsilon z - 2n^* \varepsilon^2 \leq 0.$$

Because this inequality is clearly satisfied, this completes the proof that union workers have no profitable deviation.

### **Proof of Corollary 6**

The result is implied by Corollary 2, which established that  $d\alpha^*/d\varepsilon < 0$ . This is because  $\beta$  is an increasing transformation of  $\alpha$ . Because  $\alpha N_0 = Y_w = \beta N_0(N_0 + 1)/2$ , then the wage rate,  $\beta$ , can be written as  $\beta = 2\alpha/(N_0 + 1)$ .

### **Proof of Corollary 3**

To see that union members earn less in aggregate in the equilibrium of Chapter 4 (where output is observable) than Chapter 3 (where output is unobservable) note the following two facts. First, the equilibrium workers' surplus is unchanged between the chapters because the equilibrium union size and hence the workers' bargaining power is unchanged. Second, wages in Chapter 4 are increasing with skill whereas in Chapter 3 wages are constant; in other words, in Chapter 4 the wage of higher skilled non-union workers is more than the wage of lower skilled union workers. These two factors together prove the result. The only exception is when all workers are unionized in equilibrium. In that case, the union workers accrue all of the workers' surplus (which, as described, is unchanged between the chapters).

### **Proof of Lemma 3**

Combining equations (5.1) and (5.2) yields:

$$\beta n^2 = \alpha n = \frac{K(n)[N_0(N_0 + 1) - \varepsilon n(n + 1)]}{2} - \frac{\beta[N_0(N_0 + 1) - n(n + 1)]}{2}.$$

Solving this equation for  $\beta$  establishes the wage rate of non-members. After some algebra the above equality becomes:

$$2\beta n^2 = K(n)N_0(N_0 + 1) - K(n)\varepsilon n(n + 1) - \beta N_0(N_0 + 1) + \beta n^2 + \beta n$$

$$\beta [n^2 - n + N_0(N_0 + 1)] = K(n) [N_0(N_0 + 1) - \varepsilon n(n + 1)]$$

$$\beta = \frac{K(n) [N_0(N_0 + 1) - \varepsilon n(n + 1)]}{N_0(N_0 + 1) + n(n + 1)}.$$

### Proof of Proposition 7

Take the first derivative of  $\alpha = \beta n$  with respect to  $n$ , in equation (4.2), to obtain:

$$\frac{d\alpha}{dn} = \beta + n^* \frac{d\beta(n^*)}{dn} = 0.$$

By equation (5.3), this first order condition becomes:

$$\begin{aligned} K(n^*) \frac{x - \varepsilon n^*(n^* + 1)}{x + n^*(n^* - 1)} + n^* \frac{dK(n^*)}{dn} \frac{x - \varepsilon n^*(n^* + 1)}{x + n^*(n^* - 1)} \\ - n^* K(n^*) \left( \frac{2n^*\varepsilon + \varepsilon}{x + n^*(n^* - 1)} + \frac{(2n^* - 1)(x - \varepsilon n^*(n^* + 1))}{(x + n^*(n^* - 1))^2} \right) = 0. \end{aligned}$$

Isolating  $K(n^*) / (dK(n^*)/dn)$  yields the condition (5.4). If  $\beta$  is always increasing on its domain, then its unique maximal value is  $n^* = N_0$ . On the other hand, if  $\beta$  is decreasing at  $n = N_0$ , then the first order condition establishes the unique maximizer, by the Intermediate Value Theorem (which can be applied because the objective function is continuous). Because  $K(n^*)/n^* > 0$ , and  $x > \varepsilon n^*(n^* + 1)$  for every  $n^* < N_0$ , then:

$$\begin{aligned} \frac{d\beta}{dn} \geq 0 \quad \Leftrightarrow \\ \frac{\frac{dK(n^*)}{dn}}{K(n^*)/n^*} \geq \frac{\varepsilon n^* \left[ (3n^* + 2)x + n^*(n^{*2} - 2n^* - 1) \right] - x(x - n^{*2})}{n^* [x + n^*(n^* - 1)] [x - \varepsilon n^*(n^* + 1)]}. \end{aligned}$$

At the upper corner  $n^* = N_0$ :

$$\begin{aligned} \frac{d\beta}{dn} \geq 0 \quad \Leftrightarrow \\ \gamma(N_0) = \frac{\frac{dK(N_0)}{dn}}{K(N_0)/N_0} \geq \frac{\varepsilon N_0 [(3N_0 + 2)x + N_0(N_0^2 - 2N_0 - 1)] - x(x - N_0^2)}{N_0^* [x + N_0(N_0 - 1)] [x - \varepsilon N_0(N_0 + 1)]}. \end{aligned}$$

Solving for  $\varepsilon$  the inequality on the right-hand side of the above equation

leads to:

$$\varepsilon \leq \frac{2N_0\gamma(N_0) + 1}{2N_0\gamma(N_0) + 1 + \frac{2N_0(2N_0+1)}{N_0+1}}.$$

As discussed in the text, all non-union workers have the correct incentives and the union is maximizing its utility by construction. The only possible deviation is if a union member leaves the union. Of course, if this occurred, then  $n^* < N_0$ , so that there is an interior solution for the union's problem. Suppose a worker with skill  $j < n^*$  considers leaving the union while all other players are playing according to the prescribed equilibrium.

As worker with skill  $j$  moves from inside to outside the union, the workers' surplus has a net change of  $\varepsilon j K(n^* - 1)$ . Worker with skill  $j$  has no incentive to deviate if and only if  $\beta j \leq \alpha$ .

Substituting  $\alpha$  of equation (5.2), with  $n = n^*$  and  $\beta$  of equation (4.2), with  $n = n^* - 1$ :

$$\alpha \geq K(n^* - 1) \left( \frac{N_0(N_0 + 1) - \varepsilon n^*(n^* - 1)}{N_0(N_0 + 1) + (n^* - 1)(n^* - 2)} \right) j.$$

If this inequality holds for the largest possible  $j$  in the union, then it holds for all  $j < n^*$ . This highest possible  $j$  is the marginal union member,  $n^*$ . Substituting  $j = n^*$  and  $\alpha = \beta n^*$  with  $\beta$  of equation (4.2) using  $n = n^*$  results in:

$$K(n^*) \left( \frac{x - \varepsilon n^*(n^* + 1)}{x + n^*(n^* - 1)} \right) n^* \geq K(n^* - 1) \left( \frac{x - \varepsilon n^*(n^* - 1)}{x + n^{*2} - 3n^* + 2} \right) n^*.$$

After some algebra, this inequality becomes simply:

$$\begin{aligned} & \left( \frac{K(n^*)}{K(n^* - 1)} - 1 \right) (x n^{*2} - n^* x) + \frac{K(n^*)}{K(n^* - 1)} (-2n^* x + 2x) + \\ & + \frac{K(n^*) \left[ 3\varepsilon n^{*2}(n^* + 1) - 2\varepsilon n^*(n^* + 1) - \varepsilon n^*(n^* + 1)x - \varepsilon n^3(n^* + 1) \right]}{K(n^* - 1)} \geq \\ & \geq -x\varepsilon n^{*2} + x\varepsilon n^* - \varepsilon n^{*3} + \varepsilon n^{*2}. \end{aligned}$$

The right-hand side of this inequality is negative, because  $-x\varepsilon n^{*2} < x\varepsilon n^*$  and  $-\varepsilon n^{*3} < \varepsilon n^{*2}$ . This implies the inequality will hold if the right-hand side is set to zero. For simplicity, this is how we proceed. The inequality therefore becomes:

$$\begin{aligned} & \left( \frac{K(n^*)}{K(n^* - 1)} - 1 \right) (x n^{*2} - n^* x) + \frac{K(n^*)}{K(n^* - 1)} (-2n^* x + 2x) + \\ & + \frac{K(n^*) \left[ 3\varepsilon n^{*2}(n^* + 1) - 2\varepsilon n^*(n^* + 1) - \varepsilon n^*(n^* + 1)x - \varepsilon n^{*3}(n^* + 1) \right]}{K(n^* - 1)} \geq 0. \end{aligned}$$

After some algebra, this inequality becomes:

$$\begin{aligned} & \frac{(2 - 2n^*)}{(n^{*2} - n^*)} - \frac{\varepsilon n^*(n^* + 1)x + \varepsilon n^{*3}(n^* + 1)}{x(n^{*2} - n^*)} + \\ & + \frac{3\varepsilon n^{*2}(n^* + 1) - 2\varepsilon n^*(n^* + 1)}{x(n^{*2} - n^*)} \geq -1 + \frac{K(n^* - 1)}{K(n^*)}. \end{aligned}$$

Thus:

$$\begin{aligned} & \frac{K(n^*)}{K(n^* - 1)} > \frac{(n^{*2} - n^*)}{(2 - 2n^*)} + \\ & + \frac{x(n^{*2} - n^*)}{[-\varepsilon n^*(n^* + 1)x - \varepsilon n^{*3}(n^* + 1) + 3\varepsilon n^{*2}(n^* + 1) - 2\varepsilon n^*(n^* + 1)]} + 1. \end{aligned}$$

The first two terms of the inequality are negative because  $2 - 2n^* < 0$  and  $-\varepsilon n^{*3}(n^* + 1) < 3\varepsilon n^{*2}(n^* + 1)$ . As a result, the left-hand side is less than one. The right-hand side of the inequality is greater than one because  $K(n^* - 1) < K(n^*)$  by assumption (3.4). This completes the proof.

#### **Proof of Corollary 7**

Take the limit of condition (5.5) as the population of workers grows large,  $N_0 \rightarrow +\infty$ .

#### **Proof of Corollary 8**

Denote  $R_{\alpha\beta}$  as the right-hand side of the equilibrium condition (5.4) and recall that  $n^*$  is the equilibrium union size. Re-writing this equilibrium

condition yields  $0 = \gamma(n^*) - R_{\alpha\beta}(n^*)$ . To calculate the sign of  $dn^*/d\varepsilon$ , take the partial derivative of this equilibrium condition with respect to  $\varepsilon$ , yielding:

$$0 = \frac{\partial\gamma(n^*)}{\partial n} \frac{dn^*}{d\varepsilon} - \frac{\partial R_{\alpha\beta}(n^*)}{\partial n} \frac{dn^*}{d\varepsilon} - \frac{\partial R_{\alpha\beta}(n^*)}{\partial \varepsilon},$$

and

$$\begin{aligned} \frac{\partial R_{\alpha\beta}(n^*)}{\partial \varepsilon} &= \frac{\partial\gamma(n^*)}{\partial n} \frac{dn^*}{d\varepsilon} - \frac{\partial R_{\alpha\beta}(n^*)}{\partial n} \frac{dn^*}{d\varepsilon} \\ &= \frac{dn^*}{d\varepsilon} \left[ \frac{\partial\gamma(n^*)}{\partial n} - \frac{\partial R_{\alpha\beta}(n^*)}{\partial n} \right]. \end{aligned}$$

From equation (5.4) it is clear that  $\partial R_{\alpha\beta}/\partial \varepsilon > 0$ ; in other words, both sides of the above equality are positive. Therefore, on the right-hand side of the above equality,  $dn^*/d\varepsilon < 0$  if and only if the term in the square brackets is negative. Because  $\partial R_{\alpha\beta}(n^*)/\partial n > 0$ , (by Lemma 4) the term in the square brackets is negative when  $\partial\gamma(n^*)/\partial n \leq \partial R_{\alpha\beta}(n^*)/\partial n$ . We do not know  $\partial\gamma(n^*)/\partial n$ ; if the elasticity is constant or decreasing in union size, then  $dn^*/d\varepsilon < 0$ . And if it is increasing in the union size, then  $dn^*/d\varepsilon < 0$  will still hold so long as the inequality described holds; that is, so long as  $\partial\gamma(n^*)/\partial n \leq \partial R_{\alpha\beta}(n^*)/\partial n$ . We took the derivative of equation (5.4) to  $n$  to gain more insight into the right-hand side of this inequality, but this derivative turned out to be large and not intuitive.

### Proof of Proposition 8

Let the coalition be of two or more workers;  $|\zeta| \geq 2$ . This coalition can contain unionized and non-unionized workers, formally:

$$j_U \in \zeta,$$

$$j_N \in \zeta.$$

In particular, there are three possible types of coalitions: (a) a coalition of entirely non-union workers, (b) a coalition of entirely union workers, and (c) a coalition containing both union and non-union workers.

First, consider the case where the coalition consists of non-unionized workers only, so that:

$$\zeta \subset \{\text{non-unionized workers}\}.$$

In this case, none of these workers have incentives to join the union, because they receive at least  $\beta n^*$  out of the union and  $\alpha = \beta n^*$  in the union.

Second, consider the case where the coalition consists of unionized workers only. In particular, suppose there are a total of  $c$  union members in the coalition, where  $c \leq n^*$ , so that:

$$\zeta \subset \{\text{union members}\} = \{n^*, n^* - 1, \dots, n^* - c\}$$

$$|\zeta| = c.$$

If this coalition leaves the union, the highest paid worker of the coalition will be the highest skilled, since the wage rate out of the union is increasing linearly with skill. Consequently, only the incentive condition of the marginal union member requires checking. Specifically, worker with skill  $n^*$  has no incentive to deviate with any coalition of union members if and only if  $\beta n^* \leq \alpha$ .

Substituting  $\alpha$  of equation (5.2), with  $n = n^*$  and  $\beta$  of equation (4.2), with  $n = n^* - 1$  leads to:

$$\alpha \geq K(n^* - c) \left( \frac{x - \varepsilon(n^* - c)((n^* + 1 - c))}{x + (n^* - c)(n^* - c - 1)} \right) n^*.$$

Substituting  $\alpha = \beta n^*$  with  $\beta$  of equation (4.2) using  $n = n^*$  results in:

$$K(n^*) \left( \frac{x - \varepsilon n^*(n^* + 1)}{x + n^*(n^* - 1)} \right) n^* \geq K(n^* - c) \left( \frac{x - \varepsilon(n^* - c)(n^* + 1 - c)}{x + (n^* - c)(n^* - c - 1)} \right) n^*.$$

After some algebra, this inequality becomes:

$$\begin{aligned}
& \frac{K(n^*)[x^2 + xn^2 - xn - 2ncx + cx + xc^2 + \varepsilon(-n^*(n^* + 1)x - n^3(n^* + 1))]}{K(n^* - c)} + \\
& + \frac{K(n^*)\varepsilon[n^2(n^* + 1) + 2cn^2(n^* + 1) - n^*(n^* + 1)c - n^*(n^* + 1)c^2]}{K(n^* - c)} \geq \\
& \geq x^2 + xn^*(n^* - 1) + \\
& + \varepsilon[-xn^{*2} - xn^* + x2n^*c + xc - xc^2 - n^{*3}(n^* - 1)] + \\
& + \varepsilon[-n^{*2}(n^* - 1) + 2cn^{*2}(n^* - 1) + n^*(n^* - 1)c - c^2n^*(n^* - 1)].
\end{aligned}$$

Equivalently:

$$\begin{aligned}
& \left( \frac{K(n^*)}{K(n^* - c)} - 1 \right) (x^2 + xn^{*2} - xn^*) + \frac{K(n^*)}{K(n^* - c)} (-2n^*cx + cx + xc^2) + \\
& + \frac{K(n^*)\varepsilon \left[ -n^*(n^* + 1)x - n^{*3}(n^* + 1) + n^{*2}(n^* + 1) \right]}{K(n^* - c)} + \\
& + \frac{K(n^*)\varepsilon \left[ 2cn^{*2}(n^* + 1) - n^*(n^* + 1)c - n^*(n^* + 1)c^2 \right]}{K(n^* - c)} \geq \\
& \geq \varepsilon \left( -xn^{*2} - xn^* + x2n^*c + xc - xc^2 - n^{*3}(n^* - 1) \right) + \\
& + \varepsilon[-n^{*2}(n^* - 1) + 2cn^{*2}(n^* - 1) + n^*(n^* - 1)c - c^2n^*(n^* - 1)].
\end{aligned}$$

The right-hand side of this inequality is negative because  $x2n^*c < xn^{*2}$ ,  $xc < xn^{*2}$ ,  $2cn^{*2}(n^* - 1) < n^{*3}(n^* - 1)$  and  $n^*(n^* - 1)c < c^2n^*(n^* - 1)$ . Because the right-hand side of this inequality is negative we proceed by setting it to zero and checking whether the inequality holds. Of course, if the inequality holds with a right-hand side of zero, it holds when the right-hand side is negative. Define auxiliary variables:

$$\begin{aligned}
\widehat{T}_1 &= -n^*(n^* + 1)x - n^{*3}(n^* + 1) + n^{*2}(n^* + 1), \\
\widehat{T}_2 &= 2cn^{*2}(n^* + 1) - n^*(n^* + 1)c - n^*(n^* + 1)c^2.
\end{aligned}$$

Mathematically, we check:

$$\left(\frac{K(n^*)}{K(n^* - c)} - 1\right)(x^2 + xn^{*2} - xn^*) + \frac{K(n^*)}{K(n^* - c)}(-2n^*cx + cx + xc^2) + \frac{K(n^*)}{K(n^* - c)}\varepsilon(\widehat{T}_1 + \widehat{T}_2) \geq 0.$$

After some algebra, this becomes:

$$-2n^*cx + cx + xc^2 + \varepsilon[\widehat{T}_1 + \widehat{T}_2] \geq \left(\frac{K(n^* - c)}{K(n^*)} - 1\right)(x^2 + xn^{*2} - xn^*).$$

Thus:

$$\frac{x^2 + xn^{*2} - xn^*}{-2n^*cx + cx + xc^2 + \varepsilon(\widehat{T}_1 + \widehat{T}_2)} + 1 \leq \frac{K(n^*)}{K(n^* - c)}.$$

Because the right-hand side of this inequality is greater than one, this inequality holds if the first term on the left-hand side is negative. Mathematically, we require that  $-2n^*cx + cx + xc^2 + \varepsilon T < 0$ , where:

$$T = \widehat{T}_1 + \widehat{T}_2.$$

To see that inequality  $-2n^*cx + cx + xc^2 + \varepsilon T < 0$  holds, note that  $-2n^*cx + cx + xc^2 < 0$ , as  $-2n^* + 1 + c < 0$ , and  $T < 0$ , because  $n^{*3}(n^* + 1) > n^{*2}(n^* + 1)$  and  $2cn^{*2}(n^* + 1) < n^*(n^* + 1)x$ .

Finally, consider the case where the coalition consists of both unionized and non-unionized workers. In this case, if a non-unionized worker with skill  $j > n^*$  joins the union and a unionized worker with skill  $j \leq n^*$  leaves the union, the workers' bargaining power is unchanged, but total surplus falls and hence wages of non-union workers decline. Hence, there is no profitable coalitional deviation of this form.

#### **Proof of Remark 4**

To see that  $R = -E_{Y,n}$ , note that from equation (3.7) the surplus,  $Y$ , may be written as:

$$Y = \frac{x - \varepsilon n(n + 1)}{2}.$$



Taking the derivative of this equation with respect to  $n$  yields:

$$\frac{dY}{dn} = -\varepsilon n - \frac{\varepsilon}{2}.$$

Multiplying  $dY/dn$  by  $n/Y$  yields the right-hand side of auxiliary function  $R$  and is clearly the elasticity  $-E_{Y,n}$ . To see that  $R_{\alpha\beta} = -E_{Y,n} - (x - n^2)/(x + n(n - 1))$ , first note that the union's first order condition (from Proposition 7) can be written as:

$$-\beta = n \frac{d\beta}{dn}.$$

Multiplying both sides of this equality by  $[x + n(n - 1)]^2$  yields:

$$-\beta[x + n(n - 1)]^2 = [x + n^2 - n]^2 n \frac{d\beta}{dn}.$$

In the next step, we substitute  $\beta$  from equation (5.1) and  $d\beta/dn$ , which is:

$$\frac{d\beta}{dn} = 2 \frac{[x + n(n - 1)] [KY_n + K_n Y] - KY [2n - 1]}{[x + n(n - 1)]^2}.$$

Making these substitutions into the equality yields:

$$-2KY[x + n^2 - n] = 2n [(x + n(n - 1))(KY_n + K_n Y) - KY(2n - 1)].$$

After some algebra:

$$KY (-x - n^2 + n) + KY (2n^2 - n) = n(x + n(n - 1))(KY_n + K_n Y)$$

$$KY (-x + n^2) = n(x + n(n - 1))(KY_n + K_n Y)$$

$$n^2 - x = n(x + n(n - 1)) \left( \frac{Y_n}{Y} + \frac{K_n}{K} \right).$$

Re-writing in terms of the elasticity yields:

$$E_{K,n} + E_{Y,n} = \frac{n^2 - x}{x + n(n - 1)}.$$

This proves the remark.

**Proof of Lemma 4**

The numerator of  $R$  (given by equation 3.11) is  $\varepsilon n(2n+1)$ , which clearly increases with  $n$ . The denominator of  $R$  is  $x - \varepsilon n(n+1)$ , which clearly decreases with  $n$ . Hence, it is easy to conclude that  $R(n)$  is a strictly increasing function. Define  $\widehat{R}_{\alpha\beta}$  by:

$$\begin{aligned}\widehat{R}_{\alpha\beta} &= \frac{\varepsilon n [(3n+2)x + n(n^2 - 2n - 1)] - x(x - n^2)}{x + n(n-1)} \\ &= \frac{\varepsilon n^4 - 2\varepsilon n^3 + [3\varepsilon x - \varepsilon + x]n^2 + 2\varepsilon xn - x^2}{n^2 - n + x}.\end{aligned}$$

Then, we claim that  $\widehat{R}_{\alpha\beta}$  is increasing with  $n$ . This implies that  $R_{\alpha\beta} = \widehat{R}_{\alpha\beta}[x - \varepsilon n(n+1)]^{-1}$  is the product of two strictly increasing functions of  $n$ ; and thus, must also be a strictly increasing function of  $n$ . The derivative of  $\widehat{R}_{\alpha\beta}$  with respect to  $n$  is positive if and only if:

$$\begin{aligned}(n^2 - n + x)(4\varepsilon n^3 - 6\varepsilon n^2 + [6\varepsilon x - 2\varepsilon + 2x]n + 2\varepsilon x) &\geq \\ (2n - 1)(\varepsilon n^4 - 2\varepsilon n^3 + [3\varepsilon x - \varepsilon + x]n^2 + 2\varepsilon xn - x^2).\end{aligned}$$

Equivalently:

$$\begin{aligned}[2n - 5]\varepsilon n^4 + [4\varepsilon n - (1 + 5\varepsilon)]xn^2 + [6x - 2]\varepsilon xn + \\ + [4n - 1]x^2 \geq -\varepsilon(4n^3 + n^2 + 2x^2).\end{aligned}$$

For sufficiently large  $N_0$  the left-hand side of the inequality is positive and the right-hand side negative.

**Proof of Lemma 5**

Some algebra indicates that:

$$\frac{R_{\alpha\beta}}{R} = \frac{\varepsilon n [(3n+2)x + n(n^2 - 2n - 1)] - x(x - n^2)}{\varepsilon n(2n+1) [x + n(n-1)]}.$$

Hence,  $R > R_{\alpha\beta}$  if and only if:

$$\varepsilon n[(3n+2)x + n(n^2 - 2n - 1)] - x(x - n^2) < \varepsilon n(2n+1)[x + n(n-1)].$$

Equivalently:

$$N_0(N_0 + 1) > \varepsilon n(n + 1).$$

Because  $n \leq N_0$ , this last inequality holds for every  $\varepsilon < 1$ , proving this lemma.

### **Proof of Proposition 9**

This is implied by inequality (5.8).

### **Proof of Corollary 9**

The model with more union members (which by Proposition 9 is the model which offers different wage contracts to union versus non-union workers) will have more workers affected by the union inefficiency and, hence, have a lower total surplus.

Furthermore, with relatively more union members, the workers in this model have higher bargaining power because of the assumption that  $dK/dn > 0$ . Hence, workers in this model will gain a greater share of the total surplus compared to the other models.

To see that the workers' surplus is lower in the model which offers two different wage schedules, note that in the models which offer the same wage schedule to all workers, the union is maximizing the workers' surplus by essentially choosing the union size. In other words, the union size in the models which offer the same wage schedule to all workers is the union size that maximizes the workers' surplus. Because the union size in the model that offers two wage schedules is different to this, the workers' surplus must be lower.

### **Proof of Corollary 10**

Corollary 10 is proven by combining two results established earlier. First, the equilibrium workers' surplus is relatively lower in the model that offers two wage schedules compared to the model where all workers receive  $\alpha$  (see

Corollary 9). Second, in the model that offers two wage schedules, non-union workers earn more than union workers (by construction); this compares to the model of Chapter 3 where all workers receive a constant wage. These two factors together imply that union workers must earn less in equilibrium in the model that offers two wage schedules (Chapter 5) compared to the model which offers  $\alpha$  to all workers (Chapter 3).

### Proof of Proposition 10

The firm maximizes its profit function, given by equation (6.12), by choosing the amount of costly bargaining effort,  $c$ . The first and second partial derivatives of this profit function with respect to the costly effort are, respectively:

$$\begin{aligned}\frac{\partial \pi}{\partial c} &= -Y(n) \frac{\partial K}{\partial c}(c, n) - 1, \\ \frac{\partial^2 \pi}{\partial c^2} &= -Y(n) \frac{\partial^2 K}{\partial c^2}(c, n).\end{aligned}$$

The second derivative is always negative because of assumption (6.7) that  $\partial^2 K / \partial c^2 > 0$ . The first order condition is  $\partial \pi / \partial c = 0$ . Then:

$$\begin{aligned}-Y(n) \frac{\partial K}{\partial c}(c, n) - 1 &= 0 \\ \frac{\partial K}{\partial c}(c, n) &= \frac{-1}{Y(n)}.\end{aligned}$$

By combining this first order condition with assumption (6.8), it is clear that  $\partial \pi / \partial c > 0$ , if  $c = 0$  (because the assumption is that  $\partial K(0, n) / \partial c < -1/Y(n)$ ).

Similarly, we need to show that for sufficiently large  $c$ ;  $\partial \pi / \partial c < 0$ . By assumption (6.9) we know there will be a sufficiently large  $c'$  such that  $\partial K(c', n) / \partial c > -1/Y(n)$ , but this is equivalent to  $Y(n)(\partial K / \partial c) > -1 \Leftrightarrow -Y(n)(\partial K / \partial c) - 1 < 0$ , which implies  $\partial \pi / \partial c < 0$ . By the Intermediate Value Theorem, which can be applied because the function taking each  $c$  into  $\partial K(c, n) / \partial c + 1/Y(n)$  is continuous, there exists a unique  $c^{BR} = c^{BR}(n)$  such that  $\partial K(c^{BR}, n) / \partial c = -1/Y(n)$ . The point  $c^{BR}$  maximizes the profit

and satisfies the first order condition  $\partial\pi/\partial c(c^{BR}, n) = 0$ .

### **Proof of Proposition 11**

Equation (6.16) which defines the best reply of the union, can be written as:

$$K(c, n) = \frac{x - \varepsilon n(n + 1)}{\varepsilon(2n + 1)} \frac{\partial K}{\partial n}(c, n). \quad (\text{A.12})$$

To prove  $n^{BR}$  exists and is unique we show that (a) the left-hand side of equation (A.12) is increasing with  $n$ , (b) the right-hand side of equation (A.12) is decreasing in  $n$ , and (c) there is an intersection of the right and left-hand sides of equation (A.12).

For any fixed  $c$ , the left-hand side of equation (A.12) is increasing with  $n$  because of assumption (6.3) that  $\partial K(c, n')/\partial n > 0$ . The right-hand side of equation (A.12) is decreasing in  $n$  because of assumption (6.6) that  $\partial^2 K(c, n')/\partial n^2 \leq 0$  (this ensures  $\partial K(c, n)/\partial n$  is decreasing in  $n$ ).

Now that we know the left-hand side of equation (A.12) is increasing in  $n$  and the right-hand side is decreasing in  $n$ , we prove there exists a unique  $n^*$  by showing these functions intersect. We do this by showing that, when  $n$  is sufficiently low, the left-hand side of equation (A.12) (which is increasing with  $n$ ) is below the right-hand side of equation (A.12) (which is decreasing in  $n$ ). Similarly, we show that the opposite is true when  $n$  is large; that is, when  $n$  is sufficiently large, the left-hand side of equation (A.12) is above the right-hand side of equation (A.12). Mathematically, we prove the following two inequalities hold:

$$K(c, 0) < \frac{x}{\varepsilon} \frac{\partial K}{\partial n}(c, 0), \quad (\text{A.13})$$

$$K(c, N_0) > \frac{(1 - \varepsilon)x}{\varepsilon(2N_0 + 1)} \frac{\partial K}{\partial n}(c, N_0). \quad (\text{A.14})$$

These inequalities are satisfied because inequality (A.13) is the same as assumption (6.10) and inequality (A.14) is implied by assumption (6.4).

### **Proof of Lemma 6**

#### *I. Calculating $dn^{BR}/dc$*

We know that  $\alpha N_0 = K(c, n)Y(n)$ . The first order condition of the

union's problem, for each fixed  $c \geq 0$ , leads to the optimal number of workers in the union, denoted  $n^{BR} = n^{BR}(c)$ :

$$K_n(c, n^{BR}(c))Y(n^{BR}(c)) = -K(c, n^{BR}(c))Y_n(n^{BR}(c)). \quad (\text{A.15})$$

Taking the derivative with respect to  $c$  of both sides of equation (A.15) yields:

$$\begin{aligned} & Y(n^{BR})K_{cn}(c, n^{BR}) + Y(n^{BR})K_{nn}(c, n^{BR})\frac{dn^{BR}}{dc}(c) \\ &= -Y_n(n^{BR})\left[K_c(c, n^{BR}) + K_n(c, n^{BR})\frac{dn^{BR}}{dc}(c)\right]. \end{aligned}$$

With some algebra:

$$\begin{aligned} & Y(n^{BR})K_{cn}(c, n^{BR}) + Y_n(n^{BR})K_c(c, n^{BR}) \\ &= -Y_n(n^{BR})K_n(c, n^{BR})\frac{dn^{BR}}{dc}(c) - Y(n^{BR})K_{nn}(c, n^{BR})\frac{dn^{BR}}{dc}(c). \end{aligned}$$

Solving for the  $dn^{BR}/dc$  yields:

$$\frac{dn^{BR}}{dc}(c) = -\frac{Y(n^{BR})K_{cn}(c, n^{BR}) + Y_n(n^{BR})K_c(c, n^{BR})}{Y_n(n^{BR})K_n(c, n^{BR}) + Y(n^{BR})K_{nn}(c, n^{BR})}.$$

## II. Calculating $dc^{BR}/dn$

Fix  $n \in [0, N_0]$  and consider the best reply of the firm,  $c^{BR} = c^{BR}(n)$ . Take the best reply of the firm given by equation (6.13) and re-arrange for  $Y(n)K_c(c^{BR}, n) = -1$ . Taking the derivative of both sides of this equation with respect to  $n$  yields:

$$Y(n)\left[K_{cc}(c^{BR}, n)\frac{dc^{BR}}{dn}(n) + K_{cn}\right] + Y_n(n)K_c(c^{BR}, n) = 0,$$

or equivalently:

$$Y(n)K_{cc}(c^{BR}, n)\frac{dc^{BR}}{dn}(n) + Y(n)K_{cn}(c^{BR}, n) = -Y_n(n)K_c(c^{BR}, n).$$

Solving this last equation for  $dc^{BR}/dn$  yields:

$$\frac{dc^{BR}}{dn}(n) = \frac{Y(n)K_{cn}(c^{BR}, n) + Y_n(n)K_c(c^{BR}, n)}{-Y(n)K_{cc}(c^{BR}, n)}.$$

*III. Result that  $dn^{BR}/dc$  and  $dc^{BR}/dn$  have Different Signs at Nash Equilibria*

Now, suppose  $(c^*, n^*)$  is a Nash equilibrium. This means that  $c^* = c^{BR}(n^*)$  and  $n^* = n^{BR}(c^*)$ .

From the previous calculations, it is clear that  $dn^{BR}(c)/dc = 0$  if and only if  $Y(n^{BR})K_{cn}(c, n^{BR}) + Y_n(n^{BR})K_c(c, n^{BR}) = 0$ , while  $dc^{BR}(n)/dn = 0$  if and only if  $Y(n)K_{cn}(c^{BR}, n) + Y_n(n)K_c(c^{BR}, n) = 0$ . Applying these results in the Nash equilibrium  $(c^*, n^*)$  results in the following equivalences:

$$\frac{dn^{BR}}{dc}(c^*) = 0 \Leftrightarrow K_{cn}(c^*, n^*) = \frac{-Y_n(n^*)K_c(c^*, n^*)}{Y(n^*)} \Leftrightarrow \frac{dc^{BR}}{dn}(n^*) = 0.$$

Next, suppose that  $Y(n^*)K_{cn}(c^*, n^*) \neq -Y_n(n^*)K_c(c^*, n^*)$ . To see that  $dn^{BR}(c^*)/dc$  and  $dc^{BR}(n^*)/dn$  always have opposite signs, multiply these derivatives to find:

$$\frac{dn^{BR}}{dc}(c^*) \frac{dc^{BR}}{dn}(n^*) = \frac{[Y(n^*)K_{cn}(c^*, n^*) + Y_n(n^*)K_c(c^*, n^*)]^2}{Y(n^*)K_{cc}(c^*, n^*)[Y_n(n^*)K_n(c^*, n^*) + Y(n^*)K_{nn}(c^*, n^*)]} \quad (\text{A.16})$$

As  $Y(n^*)K_{cn}(c^*, n^*) + Y_n(n^*)K_c(c^*, n^*) \neq 0$ , the numerator of the fraction on the right-hand side of equation (A.16) is positive. By assumption,  $Y > 0$ ,  $K_{cc} > 0$ , by assumption (6.7). Also,  $Y_n < 0$ ,  $K_n > 0$ , by (6.3), and  $K_{nn} < 0$ , by assumption (6.6). Hence,  $Y(n^*)K_{cc}(c^*, n^*) > 0$  and  $Y_n(n^*)K_n(c^*, n^*) + Y(n^*)K_{nn}(c^*, n^*) < 0$ . Thus:

$$Y(n^*)K_{cc}(c^*, n^*)[Y_n(n^*)K_n(c^*, n^*) + Y(n^*)K_{nn}(c^*, n^*)] < 0,$$

making the denominator of the fraction on the right-hand side of equation (A.16) to be negative. Hence, if  $Y(n^*)K_{cn}(c^*, n^*) \neq -Y_n(n^*)K_c(c^*, n^*)$ , then

$dn^{BR}(c^*)/dc$  and  $dc^{BR}(n^*)/dn$  have opposite signs:

$$\frac{dn^{BR}}{dc}(c^*) \frac{dc^{BR}}{dn}(n^*) < 0.$$

Moreover, the sign of the common numerator of each  $dn^{BR}(c^*)/dc$  and  $dc^{BR}(n^*)/dn$  depends on the cross derivative  $K_{cn}(c^*, n^*)$ . For example, the common numerator will be positive if the cross derivative is sufficiently large  $K_{cn}(c^*, n^*) > -Y_n(n^*)K_c(c^*, n^*)/Y(n^*)$ , and negative if  $K_{cn}(c^*, n^*) < -Y_n(n^*)K_c(c^*, n^*)/Y(n^*)$ . Thus:

$$\begin{aligned} \frac{dn^{BR}}{dc}(c^*) > 0 &\Leftrightarrow K_{cn}(c^*, n^*) > \frac{-Y_n(n^*)K_c(c^*, n^*)}{Y(n^*)} \Leftrightarrow \frac{dc^{BR}}{dn}(n^*) < 0 \\ \frac{dn^{BR}}{dc}(c^*) < 0 &\Leftrightarrow K_{cn}(c^*, n^*) < \frac{-Y_n(n^*)K_c(c^*, n^*)}{Y(n^*)} \Leftrightarrow \frac{dc^{BR}}{dn}(n^*) > 0. \end{aligned}$$

## Proof of Proposition 12

### I. Existence of a Nash Equilibrium

The argument for existence uses the Intermediate Value Theorem (IVT). Define an auxiliary function  $A : [0, N_0] \rightarrow \mathbb{R}$  as follows. For each  $n \in [0, N_0]$ , let  $A(n) = n^{BR}(c^{BR}(n)) - n$ . This auxiliary function is continuous because both best replies are continuous. Also,  $A(0) \geq 0$  and  $A(N_0) \leq 0$ . We claim that there is some  $n^* \in [0, N_0]$  such that  $A(n^*) = 0$ . If  $A(0) = 0$  or  $A(N_0) = 0$ , then we are done. Suppose that  $A(0) > 0$  and  $A(N_0) < 0$ . In this case, by the IVT, there is some  $n^* \in (0, N_0)$  such that  $A(n^*) = 0$ ; that is,  $n^{BR}(c^{BR}(n^*)) = n^*$ . Clearly,  $n^* \in (0, N_0)$  implies  $n^* \in [0, N_0]$ . Let  $c^* = c^{BR}(n^*)$ . Then,  $n^{BR}(c^*) = n^{BR}(c^{BR}(n^*)) = n^*$ . Because  $c^* = c^{BR}(n^*)$  and  $n^{BR}(c^*) = n^*$ , then  $(c^*, n^*)$  is a Nash equilibrium.

### II. Uniqueness of the Nash Equilibrium

The best replies are continuously differentiable functions. Clearly, we can restrict the domain of  $n^{BR}$  to the closed and bounded interval:

$$[0, \max\{c^{BR}(n) | 0 \leq n \leq N_0\}].$$



There can be no Nash equilibrium  $(c^*, n^*)$  with  $c^* > \max\{c^{BR}(n) | 0 \leq n \leq N_0\}$ . With this small technical modification, both best replies have closed and bounded domains. As best replies are defined in a closed and bounded domain, their derivatives are bounded, what mathematicians call Lipschitz continuous functions. There is a positive, globally minimum distance between any two different Nash equilibria. This implies that there are only finitely many Nash equilibria.

Suppose that there are two different Nash equilibria,  $(c_1^*, n_1^*) \neq (c_2^*, n_2^*)$ . We claim that  $c_1^* \neq c_2^*$ . If we had  $c_1^* = c_2^*$ , then  $n_1^* = n^{BR}(c_1^*) = n^{BR}(c_2^*) = n_2^*$ , and then,  $(c_1^*, n_1^*) = (c_2^*, n_2^*)$ , contradicting  $(c_1^*, n_1^*) \neq (c_2^*, n_2^*)$ . Similarly, we claim that  $n_1^* \neq n_2^*$ . This must be true, otherwise  $c_1^* = c^{BR}(n_1^*) = c^{BR}(n_2^*) = c_2^*$ , which is clearly a contradiction, as  $c_1^* \neq c_2^*$ .

Assume, without loss of generality, that there is no other Nash equilibrium with  $n$ -coordinate between  $n_1^*$  and  $n_2^*$ , and  $n_1^* < n_2^*$ . This is possible because there are at most finitely many Nash equilibria. Restrict the domain of the auxiliary function  $A(n) = n^{BR}(c^{BR}(n)) - n$  to the closed interval  $[n_1^* + \delta, n_2^* - \delta]$  for some positive  $\delta > 0$  which is sufficiently small. By the previous lemma, the derivative of function  $A$ , calculated at a Nash equilibrium  $(c^{BR}(n^*), n^*)$ , denoted  $A'(n^*)$ , is negative:

$$A'(n^*) = \frac{dn^{BR}}{dc}(c^{BR}(n^*)) \frac{dc^{BR}}{dn}(n^*) - 1 \leq -1 < 0.$$

Then, the first order Taylor approximations of  $A$  a little above  $n_1^*$  and a little below  $n_2^*$  are:  $A(n_1^* + \delta) \approx A(n_1^*) + A'(n_1^*)\delta$  and  $A(n_2^* - \delta) \approx A(n_2^*) - A'(n_2^*)\delta$ . When  $\delta > 0$  the inequality  $A' < 0$  implies that  $A(n_1^*) + A'(n_1^*)\delta < A(n_1^*)$  and  $A(n_2^*) - A'(n_2^*)\delta > A(n_2^*)$ . Pick  $\delta > 0$  sufficiently small such that  $A(n_1^* + \delta) < A(n_1^*)$  and  $A(n_2^* - \delta) > A(n_2^*)$ . Because  $(c_1^*, n_1^*)$  and  $(c_2^*, n_2^*)$  are Nash equilibria, then  $A(n_1^*) = 0$  and  $A(n_2^*) = 0$ . Thus,  $A(n_1^* + \delta) < 0$  and  $A(n_2^* - \delta) > 0$ . Because function  $A$  is continuous,  $A(n_1^* + \delta) < 0$  and  $A(n_2^* - \delta) > 0$ , then it is possible to apply again the IVT to the restriction of function  $A$  to  $[n_1^* + \delta, n_2^* - \delta]$  to find a number  $\hat{n}$ , with  $n_1^* + \delta < \hat{n} < n_2^* - \delta$

and  $A(\hat{n}) = 0$ ; that is,  $n^{BR}(c^{BR}(\hat{n})) = \hat{n}$ . Let  $\hat{c} = c^{BR}(\hat{n})$ . Clearly,  $n^{BR}(\hat{c}) = n^{BR}(c^{BR}(\hat{n})) = \hat{n}$ . Hence,  $(\hat{c}, \hat{n})$  is a Nash equilibrium that is different from both  $(c_1^*, n_1^*)$  and  $(c_2^*, n_2^*)$ . Because  $n_1^* < \hat{n} < n_2^*$ , the existence of such a Nash equilibrium contradicts the fact that there is no other Nash equilibrium with  $n$ -coordinate between  $n_1^*$  and  $n_2^*$ . This contradiction proves the uniqueness of the Nash equilibrium.

### Proof of Corollary 11

To calculate  $\alpha_{n\varepsilon}^*$ , take the partial derivative of the union's best reply with respect to the inefficiency coefficient to obtain:

$$\alpha_{n\varepsilon}^* = -\frac{1}{2N_0} (K_n n^* (n^* + 1) + K(2n^* + 1)).$$

It is clear that  $\alpha_{n\varepsilon}^* < 0$  because  $K_n > 0$  by assumption 6.3. To calculate  $\pi_{c\varepsilon}^*$ , take the partial derivative of the firm's best reply with respect to the inefficiency coefficient to obtain:

$$\pi_{c\varepsilon}^* = K_c \frac{n^* (n^* + 1)}{2N_0} - 1.$$

It is clear that  $\pi_{c\varepsilon}^* < 0$  because  $K_c < 0$  by assumption 6.5.

### Proof of Proposition 13

*I. There Exists a Unique Best Reply for the Firm*

Taking the partial derivative of equation (6.12) with respect to the firm's bargaining cost,  $c$ , (to get the marginal profit with respect to the bargaining cost) yields:

$$\frac{\partial \pi}{\partial c} = -Y(n) \frac{\partial K}{\partial c}(c, n) - 1.$$

Substituting  $\partial K / \partial c$  yields:

$$\frac{\partial \pi}{\partial c} = \frac{n^2}{(c + n^2)^2} Y(n) - 1.$$

The second order partial derivative is:

$$\frac{\partial^2 \pi}{\partial c^2} = \frac{-2n^2}{(c + n^2)^3} Y(n) < 0.$$

Hence, for every fixed  $n$ , the profit is a concave function of  $c$ . Hence, the following first order condition gives us the unique maximizer  $c^{BR}(n)$ :

$$\frac{n^2}{(c + n^2)^2} Y(n) = 1.$$

Equivalently:

$$n\sqrt{Y(n)} = c + n^2.$$

Substituting  $Y(n) = [N_0(N_0 + 1) - \varepsilon n(n + 1)]/2$  into this equation yields:

$$n\sqrt{\frac{N_0(N_0 + 1) - \varepsilon n(n + 1)}{2}} = c + n^2.$$

Solving this inequality for  $c$  yields:

$$c = n\sqrt{\frac{N_0(N_0 + 1) - \varepsilon n(n + 1)}{2}} - n^2.$$

The firm's profit is increasing at  $c = 0$ . To see this, suppose that  $\lim_{N_0 \rightarrow +\infty} n/N_0 < 1$  and  $N_0$  is sufficiently large. When  $c = 0$ , the objective function  $\pi$  is strictly increasing:

$$\sqrt{\frac{N_0(N_0 + 1) - \varepsilon n(n + 1)}{2}} > n.$$

Equivalently:

$$\frac{N_0(N_0 + 1)}{n^2} - \varepsilon \left( \frac{n + 1}{n} \right) > 2.$$

Because the function taking the bargaining cost  $c$  into the profit function  $\pi$  is strictly concave, and is strictly increasing when  $c = 0$ , the Intermediate Value Theorem ensures there exists a unique  $c^{BR}$  that solves the firm's problem (that is, there exists a unique  $c^{BR}$  such that the first order partial

derivative of the firm's profit function, with respect to  $c$ , equals zero).

*II. The Best Reply of the Firm Decreases with Union Size when  $n^2 > c$*

From Lemma 6, the best reply of the firm is:

$$\frac{dc^{BR}}{dn} = \frac{YK_{cn} + K_cY_n}{-YK_{cc}}.$$

Because the denominator  $YK_{cc}$  is negative (by assumption (6.7) that  $K_{cc} > 0$ ), this derivative is negative if the numerator is positive; that is, if  $YK_{cn} + K_cY_n \geq 0$ . This clearly occurs if the cross derivative is sufficiently large (or equivalently, when  $n^2 > c$ ).

### **Proof of Lemma 7**

Substituting the bargaining power function into the equation for the wage (equation 6.15) results in:

$$\alpha = \frac{n^2}{2N_0(c + n^2)} [N_0(N_0 + 1) - \varepsilon n(n + 1)]. \quad (\text{A.17})$$

Substituting the value of  $c = c^{BR}$  given by equation (6.24), this equation becomes:

$$\begin{aligned} \alpha &= \frac{n^2}{2N_0n\sqrt{\frac{N_0(N_0+1)-\varepsilon n(n+1)}{2}}} [N_0(N_0 + 1) - \varepsilon n(n + 1)] \\ &= \frac{\sqrt{2}}{2} \frac{n}{N_0} \sqrt{N_0(N_0 + 1) - \varepsilon n(n + 1)}. \end{aligned}$$

### **Proof of Proposition 14**

#### *I. Union Size*

Taking the partial derivative with respect to  $n$  in equation (A.17), we

find:

$$\begin{aligned}
\frac{\partial \alpha}{\partial n} &= \frac{n^2}{2N_0(c+n^2)} [-\varepsilon(2n+1)] + \frac{2nc}{2N_0(c+n^2)^2} [N_0(N_0+1) - \varepsilon n(n+1)] \\
&= \frac{-\varepsilon n^2(2n+1)(c+n^2)}{2N_0(c+n^2)^2} + \frac{2nc[N_0(N_0+1) - \varepsilon n(n+1)]}{2N_0(c+n^2)^2} \\
&= \frac{2ncN_0(N_0+1) - \varepsilon n^2[(2n+1)(c+n^2) + 2c(n+1)]}{2N_0(c+n^2)^2}. \tag{A.18}
\end{aligned}$$

This marginal rate is non-negative if and only if:

$$2cN_0(N_0+1) \geq \varepsilon n[(2n+1)(c+n^2) + 2c(n+1)].$$

Equivalently:

$$\frac{2cN_0(N_0+1)}{\varepsilon} \geq n[2n^3 + n^2 + 4cn + 3c].$$

The best reply of the union is the function that takes each  $c > 0$  and returns  $N_0$  if  $n^{BR} \geq N_0$ , or  $n^{BR}$ , the unique positive solution of the first order condition:

$$n^{BR}[2n^{BR3} + n^{BR2} + 4cn^{BR} + 3c] - \frac{2cN_0(N_0+1)}{\varepsilon} = 0.$$

Moreover,  $\partial \alpha / \partial n$  is clearly decreasing in  $n^{BR}$  (from equation A.18); that is,  $\partial^2 \alpha / \partial n^2 < 0$ .

*II. To Show that  $dn^{BR}/dc > 0$*

The result is clear after writing the equation defining the best reply of the union (equation 6.26), as:

$$\frac{n^{BR}[2n^{BR3} + n^{BR2}]}{\frac{2N_0(N_0+1)}{\varepsilon} - n^{BR}(3 + 4n^{BR})} = c.$$

The numerator on the left-hand side is increasing in  $n^{BR}$  and the denominator is decreasing in  $n^{BR}$ . Hence, if  $c$  rises so too must  $n^{BR}$  for the equality

above to hold.

### III. Full Membership

Full membership occurs if and only if  $\varepsilon$  is sufficiently small. To find how small, suppose that equation (6.26) holds at  $n = N_0$ , mathematically:

$$2N_0^4 + N_0^3 + 4cN_0^2 + 3cN_0 = \frac{2cN_0(N_0 + 1)}{\varepsilon}.$$

After some algebra, this becomes:

$$\begin{aligned} \varepsilon &= \frac{2cN_0 + 2c}{2N_0^3 + N_0^2 + 4cN_0 + 3c} \\ &= \frac{\frac{2c}{N_0^2} + \frac{2c}{N_0^3}}{2 + \frac{1}{N_0} + \frac{4c}{N_0^2} + \frac{3c}{N_0^3}}. \end{aligned}$$

Letting  $c = c^*$  and taking the limit as  $N_0$  grows large yields:

$$\lim_{N_0 \rightarrow +\infty} \varepsilon = \frac{c_\infty^*}{1 + 2c_\infty^*}.$$

### Proof of Proposition 15

This is implied by Proposition 12; the proof of Proposition 12 did not depend on the bargaining function. The proof of Proposition 12 relied on the fact that  $(dn^{BR}/dc)(dc^{BR}/dn) \leq 0$ . This is also true in this particular case when  $n^2 > c$ . This is because Proposition 14 proved that  $dn^{BR}/dc > 0$  and Proposition 13 proved that  $dc^{BR}/dn < 0$  when the cross derivative  $K_{cn}$  was positive, or equivalently when  $n^2 > c$ . It also showed that  $dc^{BR}/dn < 0$  when  $N_0$  was large.

### Proof of Proposition 16

#### I. Equilibrium Condition

The best reply of the union, given by equation (6.26), can be rewritten

as:

$$\begin{aligned} \frac{n^{BR}}{N_0} \left[ 2 \left( \frac{n^{BR}}{N_0} \right)^3 + \frac{1}{N_0} \left( \frac{n^{BR}}{N_0} \right)^2 + 4 \left( \frac{c}{N_0^2} \right) \frac{n^{BR}}{N_0} + \frac{3}{N_0} \frac{c}{N_0^2} \right] = \\ = \frac{2}{\varepsilon} \left( \frac{c}{N_0^2} \right) \left( \frac{N_0 + 1}{N_0} \right). \end{aligned}$$

Letting  $c = c^*$  and  $n^{BR} = n^*$  and taking the limit as the workforce size grows large,  $N_0 \rightarrow +\infty$ , on both sides of the previous equation results in:

$$\eta_\infty^* \left[ 2\eta_\infty^{*3} + 4c_\infty^* \eta_\infty^* \right] = \frac{2c_\infty^*}{\varepsilon}.$$

Solving this equation for  $c_\infty^*$  results in:

$$2\eta_\infty^{*4} = \frac{2c_\infty^*}{\varepsilon} - 4c_\infty^{*2} \eta_\infty^*.$$

Or, equivalently, as given by equation (6.27):

$$c_\infty^* = \frac{\eta_\infty^{*4}}{\varepsilon^{-1} - 2\eta_\infty^*}.$$

The reply of the firm, given by equation (6.24), can be rewritten as:

$$\frac{c^{BR}}{N_0^2} = \frac{1}{\sqrt{2}} \frac{n}{N_0} \sqrt{\frac{N_0 + 1}{N_0}} - \varepsilon \frac{n}{N_0} \frac{n + 1}{N_0} - \left( \frac{n}{N_0} \right)^2.$$

Letting  $c^{BR} = c^*$  and  $n = n^*$  and taking the limit as the workforce size grows large,  $N_0 \rightarrow +\infty$ , on both sides of this equation results in equation (6.28):

$$c_\infty^* = \frac{\eta_\infty^*}{\sqrt{2}} \sqrt{1 - \varepsilon \eta_\infty^{*2}} - \eta_\infty^{*2}.$$

Combining equations (6.27) and (6.28) yields:

$$\frac{\eta_\infty^{*4}}{\varepsilon^{-1} - 2\eta_\infty^{*2}} = \frac{\eta_\infty^*}{\sqrt{2}} \sqrt{1 - \varepsilon \eta_\infty^{*2}} - \eta_\infty^{*2}.$$

Thus:

$$\begin{aligned}\varepsilon^{-1}\eta_{\infty}^*\sqrt{1-\varepsilon\eta_{\infty}^{*2}} - \sqrt{2}\varepsilon^{-1}\eta_{\infty}^{*2} - 2\eta_{\infty}^{*3}\sqrt{1-\varepsilon\eta_{\infty}^{*2}} + 2\sqrt{2}\eta_{\infty}^{*4} &= \sqrt{2}\eta_{\infty}^{*4} \\ \varepsilon\eta_{\infty}^{*4} - \eta_{\infty}^{*2} + \frac{1}{2+4\varepsilon} &= 0.\end{aligned}$$

Let  $z = \eta^2$ . Then:

$$\varepsilon z^2 - z + \frac{1}{2+4\varepsilon} = 0.$$

The roots of this quadratic are:

$$z_+ = \frac{1 + \sqrt{1 - \frac{2\varepsilon}{1+2\varepsilon}}}{2\varepsilon}, \quad z_- = \frac{1 - \sqrt{1 - \frac{2\varepsilon}{1+2\varepsilon}}}{2\varepsilon}.$$

It is easy to prove that  $z_+ > 1$ , and so this root cannot correspond to a solution because  $\eta^2 < 1$ . After some algebra, it is possible to prove that:

$$\begin{aligned}z_- &= \frac{1 - \sqrt{\frac{1}{1+2\varepsilon}}}{2\varepsilon} \times \frac{1 + \sqrt{\frac{1}{1+2\varepsilon}}}{1 + \sqrt{\frac{1}{1+2\varepsilon}}}, \\ z_- &= \left(1 + 2\varepsilon + \sqrt{1+2\varepsilon}\right)^{-1}.\end{aligned}$$

Hence:

$$\begin{aligned}\eta_{\infty}^* &= \sqrt{z_-} \\ &= \left(1 + 2\varepsilon + \sqrt{1+2\varepsilon}\right)^{-1/2}.\end{aligned}$$

### III. The Firm's Costly Effort is Positive in Equilibrium

The ratio of the firm's costly effort to the workforce size is given by equation (6.28) (copied below):

$$c_{\infty}^* = \frac{\eta_{\infty}^*}{\sqrt{2}}\sqrt{1 - \varepsilon\eta_{\infty}^{*2}} - \eta_{\infty}^{*2}.$$



Hence, the firm's costly effort is positive if:

$$\frac{\eta_{\infty}^*}{\sqrt{2}} \sqrt{1 - \varepsilon \eta_{\infty}^{*2}} > \eta_{\infty}^{*2}$$

$$\sqrt{1 - \varepsilon \eta_{\infty}^{*2}} > \sqrt{2} \eta_{\infty}^*$$

$$1 - \varepsilon \eta_{\infty}^{*2} > 2 \eta_{\infty}^{*2}$$

$$1 > \eta_{\infty}^{*2} (2 + \varepsilon)$$

$$\frac{1}{2 + \varepsilon} > \eta_{\infty}^{*2}.$$

Substituting the value of  $\eta_{\infty}^*$  from equation (6.29) results in:

$$\frac{1}{2 + \varepsilon} > \frac{1}{1 + 2\varepsilon + \sqrt{1 + 2\varepsilon}}$$

$$1 + 2\varepsilon + \sqrt{1 + 2\varepsilon} > 2 + \varepsilon$$

$$\sqrt{1 + 2\varepsilon} > 1 - \varepsilon.$$

This inequality is clearly satisfied, hence the firm's costly effort is always positive.

### *III. Full Union Membership*

The solution for  $\eta$  given by the two best replies in Proposition (16) are:

$$c_{\infty}^* = \frac{\eta_{\infty}^{*4}}{\varepsilon^{-1} - 2\eta_{\infty}^*} = \frac{\eta_{\infty}^*}{\sqrt{2}} \sqrt{1 - \varepsilon \eta_{\infty}^{*2}} - \eta_{\infty}^{*2}.$$

We solve for the maximum inefficiency for which all workers would be unionized (that is, if the inefficiency were more than this value, there would not be full unionization). If all workers are unionized, then  $\eta = 1$ , and the above

equality becomes:

$$\begin{aligned}\frac{1}{\varepsilon^{-1} - 2} &= \sqrt{\frac{1 - \varepsilon}{2}} - 1 \\ \frac{\varepsilon}{1 - 2\varepsilon} + \frac{1 - 2\varepsilon}{1 - 2\varepsilon} &= \sqrt{\frac{1 - \varepsilon}{2}} \\ \frac{1 - \varepsilon}{1 - 2\varepsilon} &= \sqrt{\frac{1 - \varepsilon}{2}}.\end{aligned}$$

Simplifying:

$$\sqrt{1 - \varepsilon}\sqrt{2} = 1 - 2\varepsilon.$$

Squaring both sides helps us to find the solution, but it also introduces one extra (wrong) solution. More precisely, not all solutions of the following are actually solutions of the previous equation. Squaring both sides leads to:

$$\begin{aligned}2(1 - \varepsilon) &= (1 - 2\varepsilon)^2 \\ 2 - 2\varepsilon &= 1 - 4\varepsilon + 4\varepsilon^2 \\ 4\varepsilon^2 - 2\varepsilon - 1 &= 0.\end{aligned}$$

This quadratic has two roots, which are defined by:

$$\varepsilon = \frac{1 \pm \sqrt{5}}{4}.$$

The negative of the two roots, that is,  $-0.31$ , is the only one of the two roots that satisfies the best replies in Proposition 16 (given that a positive value for  $\varepsilon$  leads to equation (6.28) taking the root of a negative number). The positive root does not satisfy the equation  $\sqrt{1 - \varepsilon}\sqrt{2} = 1 - 2\varepsilon$ , as the right-hand side becomes negative when  $\varepsilon = (1 + \sqrt{5})/4$ . Because  $\varepsilon > 0$ , this result implies that there is never complete unionization of the workforce.

### **Proof of Corollary 12**

*I. Comparative Static Result for the Equilibrium Union Size and Costly Bargaining Effort with respect to the Inefficiency Coefficient*

It is straightforward to see that  $n^*$  and  $c^*$  are decreasing in  $\varepsilon$  after establishing they both move in the same direction all the time. To see that  $n^*$  and  $c^*$  move in the same direction, note that the bargaining elasticity can be written as:

$$\begin{aligned}\frac{n^*}{K(c^*, n^*)} \frac{\partial K}{\partial n} &= \gamma_n(c^*, n^*) \\ \gamma_n \frac{n^{*2}}{n^{*2} + c^*} &= \frac{2n^* c^*}{(n^{*2} + c^*)^2} n^* \\ c^* &= \frac{\gamma_n n^{*2}}{2 - \gamma_n}.\end{aligned}\tag{A.19}$$

Because  $\gamma_n$  is constant, this equality shows that  $c^*$  and  $n^*$  are increasing functions of each other. To see that  $n^*$  and  $c^*$  are decreasing in  $\varepsilon$ , note that the equilibrium condition given by equation (6.26) in Proposition 14 can be written as:

$$\varepsilon = \frac{2x \frac{\gamma_n}{1-\gamma_n}}{2n^{*3} + n^{*2} \left(1 + 4 \frac{\gamma_n}{1-\gamma_n}\right) + 3n^* \frac{\gamma_n}{1-\gamma_n}}.$$

Clearly, because  $\gamma_n$  is constant, as  $\varepsilon$  rises,  $n^*$  must fall (because the denominator is clearly increasing in  $n^*$ ). Because  $c^*$  moves in the same direction, so too must  $c^*$ .

## II. Comparative Static Result for the Equilibrium Wage and Inefficiency Coefficient

First note that the bargaining function is a constant when the elasticity is constant. To see this, simply substitute equation (A.19) into the bargaining function as follows:

$$\begin{aligned}K &= \frac{n^{*2}}{n^{*2} + c^*} \\ &= \frac{n^{*2}}{n^{*2} \left(1 + \frac{\gamma_n}{2-\gamma_n}\right)} \\ &= \frac{1}{1 + \frac{\gamma_n}{2-\gamma_n}}.\end{aligned}\tag{A.20}$$

Next, taking the derivative of the wage  $\alpha = K(c^*, n^*)Y/(2N_0)$  with respect to  $\varepsilon$  and substituting the expression for  $Y$  results in:

$$2N_0 \frac{d\alpha^*}{d\varepsilon} = [x - \varepsilon n^*(n^* + 1)] \frac{dK(c^*, n^*)}{d\varepsilon} - K(c^*, n^*) \left( \varepsilon(2n^* + 1) \frac{dn^*}{d\varepsilon} + n^*(n^* + 1) \right).$$

Because the bargaining function is a constant and  $dK(c^*, n^*)/d\varepsilon = 0$  by equation (A.20) this equality becomes:

$$\frac{2N_0}{K(c^*, n^*)} \frac{d\alpha^*}{d\varepsilon} = -\varepsilon(2n^* + 1) \frac{dn^*}{d\varepsilon} - n^*(n^* + 1). \quad (\text{A.21})$$

Next, we substitute  $dn^*/d\varepsilon$  into this equation. To derive an expression for  $dn^*/d\varepsilon$ , first substitute equation (A.19) into equation (6.26), yielding:

$$n^{*4} \left( 2 + 4 \frac{\gamma_n}{2 - \gamma_n} \right) + n^{*3} \left( 1 + 3 \frac{\gamma_n}{2 - \gamma_n} \right) = \frac{2n^{*2}x}{\varepsilon} \frac{\gamma_n}{2 - \gamma_n}.$$

Taking the derivative of this equation with respect to  $\varepsilon$  yields:

$$\begin{aligned} 4n^{*3} \frac{dn^*}{d\varepsilon} \left( 2 + 4 \frac{\gamma_n}{2 - \gamma_n} \right) + 3n^{*2} \frac{dn^*}{d\varepsilon} \left( 1 + 3 \frac{\gamma_n}{2 - \gamma_n} \right) &= \\ &= \frac{4n^{*3}x}{\varepsilon} \frac{dn^*}{d\varepsilon} \frac{\gamma_n}{2 - \gamma_n} - \frac{2n^{*2}x}{\varepsilon^2} \frac{\gamma_n}{2 - \gamma_n}. \end{aligned}$$

Solving this equation for  $dn^*/d\varepsilon$  yields:

$$\frac{dn^*}{d\varepsilon} = \frac{-2n^{*2}x \frac{\gamma_n}{2 - \gamma_n}}{\left[ 4n^{*3} \left( 2 + 4 \frac{\gamma_n}{2 - \gamma_n} \right) + 3n^{*2} \left( 1 + 3 \frac{\gamma_n}{2 - \gamma_n} \right) - 4xn^* \frac{\gamma_n}{\varepsilon(2 - \gamma_n)} \right] \varepsilon^2}. \quad (\text{A.22})$$

Substituting this expression for  $dn^*/d\varepsilon$  into equation (A.21) yields:

$$\begin{aligned} \frac{2N_0}{K(c^*, n^*)} \frac{d\alpha^*}{d\varepsilon} &= \\ &= \frac{\varepsilon(2n^* + 1) \frac{2n^{*2} x \gamma_n}{2 - \gamma_n}}{\left[ 4n^{*3} \left( 2 + 4 \frac{\gamma_n}{2 - \gamma_n} \right) + 3n^{*2} \left( 1 + 3 \frac{\gamma_n}{2 - \gamma_n} \right) - 4xn^* \frac{\gamma_n}{\varepsilon(2 - \gamma_n)} \right] \varepsilon^2} \\ &\quad - n^*(n^* + 1). \end{aligned}$$

The derivative  $d\alpha^*/d\varepsilon$  will be negative if:

$$\begin{aligned} &\varepsilon(2n^* + 1) \left( \frac{2n^{*2} x \gamma_n}{2 - \gamma_n} \right) \\ &\frac{\left[ 4n^{*3} \left( 2 + 4 \frac{\gamma_n}{2 - \gamma_n} \right) + 3n^{*2} \left( 1 + 3 \frac{\gamma_n}{2 - \gamma_n} \right) - 4xn^* \frac{\gamma_n}{\varepsilon(2 - \gamma_n)} \right] \varepsilon^2} < n^*(n^* + 1). \end{aligned}$$

Multiplying by the denominator of  $dn^*/d\varepsilon$  (this denominator is positive because  $dn^*/d\varepsilon < 0$  and its numerator is negative) yields:

$$\begin{aligned} &\frac{\varepsilon(2n^* + 1)(2n^{*2} x \gamma_n)}{2 - \gamma_n} < \\ &< \varepsilon^2 n^*(n^* + 1) \left[ 4n^{*3} \left( 2 + \frac{4\gamma_n}{2 - \gamma_n} \right) + 3n^{*2} \left( 1 + \frac{3\gamma_n}{2 - \gamma_n} \right) - \frac{4xn^* \gamma_n}{\varepsilon(2 - \gamma_n)} \right]. \end{aligned}$$

Dividing both sides by  $\varepsilon N_0^5$  yields:

$$\begin{aligned} &\left( \frac{2n^* + 1}{N_0} \right) \left[ 2 \frac{n^{*2}}{N_0^2} \frac{x}{N_0^2} \frac{\gamma_n}{2 - \gamma_n} \right] < \\ &< \frac{\varepsilon n^*}{N_0} \left( \frac{n^*}{N_0} + \frac{1}{N_0} \right) \left[ \frac{4n^{*3}}{N_0^3} \left( 2 + \frac{4\gamma_n}{2 - \gamma_n} \right) + \frac{3n^{*2}}{N_0^3} \left( 1 + \frac{3\gamma_n}{2 - \gamma_n} \right) - \frac{4x}{N_0^2} \frac{n^*}{N_0} \frac{\gamma_n}{\varepsilon(2 - \gamma_n)} \right]. \end{aligned}$$

Taking the limit as  $N_0$  grows large yields:

$$4 \left( \frac{n^*}{N_0} \right) \left( \frac{\gamma_n}{2 - \gamma_n} \right) < \varepsilon \left[ 4 \left( \frac{n^*}{N_0} \right)^3 \left( 2 + 4 \frac{\gamma_n}{2 - \gamma_n} \right) - 4 \frac{n^*}{N_0} \frac{\gamma_n}{\varepsilon (2 - \gamma_n)} \right]$$

$$\left( \frac{\gamma_n}{2 - \gamma_n} \right) < \varepsilon \left[ \left( \frac{n^*}{N_0} \right)^2 \left( 2 + 4 \frac{\gamma_n}{2 - \gamma_n} \right) - \frac{\gamma_n}{\varepsilon (2 - \gamma_n)} \right].$$

Substituting equation (6.29) for  $n^*/N_0$  yields:

$$\left( \frac{\gamma_n}{2 - \gamma_n} \right) < \varepsilon \left[ \frac{1}{1 + 2\varepsilon + \sqrt{1 + 2\varepsilon}} \left( 2 + 4 \frac{\gamma_n}{2 - \gamma_n} \right) - \frac{\gamma_n}{\varepsilon (2 - \gamma_n)} \right]$$

$$\left( \frac{\gamma_n}{2 - \gamma_n} \right) < \frac{\varepsilon}{1 + 2\varepsilon + \sqrt{1 + 2\varepsilon}} \left( 2 + 4 \frac{\gamma_n}{2 - \gamma_n} \right) - \frac{\gamma_n}{(2 - \gamma_n)}$$

$$\frac{\gamma_n}{2 - \gamma_n} < \frac{\varepsilon}{1 + 2\varepsilon + \sqrt{1 + 2\varepsilon}} \left( 1 + 2 \left( \frac{\gamma_n}{2 - \gamma_n} \right) \right).$$

After some algebra this can be written as:

$$1 < \frac{\varepsilon}{1 + 2\varepsilon + \sqrt{1 + 2\varepsilon}} \left( \frac{1}{\frac{\gamma_n}{2 - \gamma_n}} + 2 \right)$$

$$\frac{2 - \gamma_n}{\gamma_n} + 2 > \frac{1 + 2\varepsilon + \sqrt{1 + 2\varepsilon}}{\varepsilon}$$

$$\frac{2}{\gamma_n} + 1 > \frac{1 + 2\varepsilon + \sqrt{1 + 2\varepsilon}}{\varepsilon}$$

$$\gamma_n < \frac{2}{\varepsilon} \left( 1 + \varepsilon - \sqrt{1 + 2\varepsilon} \right).$$

### III. Comparative Static Result for the Equilibrium Union Size and Firm Effort with respect to Workforce Size

To see that  $n^*$  and  $c^*$  are increasing in  $N_0$ , note that the equilibrium

condition given by equation (6.26) in Proposition 14 can be written as:

$$N_0(N_0 + 1) = \varepsilon \left( \frac{n^*(2n^* + 1)}{\gamma_n} + n^*(n^* + 1) \right).$$

Clearly, if  $N_0$  rises so too does  $n^*$ . Because  $n^*$  is rising with  $N_0$ , so too is  $c^*$  (by the result earlier in this proof that  $n^*$  and  $c^*$  are increasing functions of each other).

#### *IV. Comparative Static Result for the Equilibrium Union Density and Workforce Size*

To see that  $\lim_{N_0 \rightarrow +\infty} d(n^*/N_0)/dN_0 = 0$ , note that  $\eta_\infty^* = \lim_{N_0 \rightarrow +\infty} (n^*/N_0) = (1 + 2\varepsilon + \sqrt{1 + 2\varepsilon})^{-1/2}$  (from Proposition 16). This expression for  $\eta_\infty^*$  depends only on  $\varepsilon$ . This proves the result.

#### *V. Comparative Static Result for the Equilibrium Surplus and Inefficiency Coefficient*

Because  $K(c, n)$  is constant (a result established earlier in this proof), the ratios  $Y_w^*/Y^*$  and  $Y_f^*/Y^*$  must be constant, because they are referring to shares of the surplus. Also, note that the wage is falling in the inefficiency if the elasticity is sufficiently small (a result established earlier in the proof). Because of the Budget Balance constraint (aggregate wages equal the workers' surplus), when wages fall,  $Y_w^*$  falls. If a rise in the inefficiency causes  $Y_w^*$  to fall, then  $Y^*$  must fall because the ratio  $Y_w^*/Y^*$  is a constant. The same is true for the firm surplus; if a rise in the inefficiency causes  $Y^*$  to fall, then  $Y_f^*$  must fall because the ratio  $Y_f^*/Y^*$  is a constant.

#### *VI. Comparative Static Result for the Equilibrium Firm Profit and Inefficiency Coefficient*

The firm profit is given by  $\pi^* = (1 - K(c^*, n^*))Y(n^*) - c^*$ . Substituting equation (A.19) into this equality yields:

$$\pi^* = (1 - K(c^*, n^*))Y(n^*) - n^{*2} \left( \frac{\gamma_n}{2 - \gamma_n} \right).$$

Taking the derivative of this equation with respect to the inefficiency yields:

$$\begin{aligned}\frac{d\pi^*}{d\varepsilon} &= -\frac{(1 - K(c^*, n^*))}{2} \left( \varepsilon(2n^* + 1) \frac{dn^*}{d\varepsilon} + n^*(n^* + 1) \right) \\ &\quad - 2n^* \frac{dn^*}{d\varepsilon} \left( \frac{\gamma_n}{2 - \gamma_n} \right) \\ \frac{2}{(1 - K(c^*, n^*))} \frac{d\pi^*}{d\varepsilon} &= -\varepsilon \left( 2n^* + 1 + \frac{4\gamma_n n^*}{(2 - \gamma_n)(1 - K(c^*, n^*))\varepsilon} \right) \frac{dn^*}{d\varepsilon} \\ &\quad - n^*(n^* + 1).\end{aligned}$$

Substituting  $dn^*/d\varepsilon$  (from equation A.22) into this equation allows us to write the right-hand side as follows (the denominator of  $dn^*/d\varepsilon$  is positive because  $dn^*/d\varepsilon < 0$ , and its numerator is negative; also  $K$  is a constant as described earlier in the proof):

$$\begin{aligned}& -\varepsilon \left( 2n^* + 1 + \frac{4\gamma_n n^*}{(2 - \gamma_n)(1 - K(c^*, n^*))\varepsilon} \right) \\ & \left( \frac{-2n^{*2} x \frac{\gamma_n}{2 - \gamma_n}}{\varepsilon^2 \left( 4n^{*3} \left( 2 + 4\frac{\gamma_n}{2 - \gamma_n} \right) + 3n^{*2} \left( 1 + 3\frac{\gamma_n}{2 - \gamma_n} \right) - 4xn^* \frac{\gamma_n}{\varepsilon(2 - \gamma_n)} \right)} \right) \\ & \quad - n^*(n^* + 1).\end{aligned}$$

We are interested when this right-hand term is negative. Hence, moving the  $-n(n + 1)$  term to the right of an inequality and multiplying everything by the denominator of the second term (which is positive as described above) yields:

$$\begin{aligned}& \varepsilon \left( 2n^* + 1 + \frac{4\gamma_n n^*}{(2 - \gamma_n)(1 - K(c^*, n^*))\varepsilon} \right) \frac{2n^{*2} x \gamma_n}{2 - \gamma_n} < \\ & < n^*(n^* + 1) \varepsilon^2 \left( 4n^{*3} \left( 2 + \frac{4\gamma_n}{2 - \gamma_n} \right) + 3n^{*2} \left( 1 + \frac{3\gamma_n}{2 - \gamma_n} \right) - \frac{4xn^* \gamma_n}{\varepsilon(2 - \gamma_n)} \right).\end{aligned}$$



Dividing all terms by  $n^2 N_0^3$  yields:

$$\begin{aligned} & \varepsilon \left( 2 \frac{n^*}{N_0} + \frac{1}{N_0} + \frac{4\gamma_n n^*}{N_0 (2 - \gamma_n) (1 - K(c^*, n^*)) \varepsilon} \right) \frac{2x}{N_0^2} \frac{\gamma_n}{2 - \gamma_n} < \\ & < \left( 1 + \frac{1}{n} \right) \varepsilon^2 \left( \frac{4n^{*3}}{N_0^3} \left( 2 + \frac{4\gamma_n}{2 - \gamma_n} \right) + \frac{3n^{*2}}{N_0^3} \left( 1 + \frac{3\gamma_n}{2 - \gamma_n} \right) - \frac{4x}{N_0^2} \frac{n^*}{N_0} \frac{\gamma_n}{\varepsilon (2 - \gamma_n)} \right). \end{aligned}$$

As  $N_0$  grows large this becomes:

$$\begin{aligned} & \varepsilon \left( 2 + \frac{4\gamma_n}{(2 - \gamma_n) (1 - K(c^*, n^*)) \varepsilon} \right) \frac{n^*}{N_0} \frac{2\gamma_n}{2 - \gamma_n} < \\ & < \varepsilon^2 \left( 4 \left( 2 + 4 \frac{\gamma_n}{2 - \gamma_n} \right) - \frac{4n^*}{N_0} \frac{\gamma_n}{\varepsilon (2 - \gamma_n)} \right). \end{aligned}$$

Multiplying both sides of the inequality by  $(2 - \gamma_n) / \varepsilon$  gives:

$$\left( 2 + \frac{4\gamma_n}{(2 - \gamma_n) (1 - K(c^*, n^*)) \varepsilon} \right) \frac{2\gamma_n n^*}{N_0} < 4\varepsilon \left( 2 + 4\gamma_n - \frac{n^*}{N_0} \frac{\gamma_n}{\varepsilon} \right).$$

Gathering like terms yields:

$$\begin{aligned} & 2\gamma_n \frac{n^*}{N_0} \left( 2 + \frac{4\gamma_n}{(2 - \gamma_n) (1 - K(c^*, n^*)) \varepsilon} + 4\gamma_n \right) < 4\varepsilon (2 + 4\gamma_n) \\ & \gamma_n \frac{n^*}{N_0} \left( 1 + \frac{2\gamma_n}{(2 - \gamma_n) (1 - K(c^*, n^*)) \varepsilon} + 2\gamma_n \right) < \varepsilon (2 + 4\gamma_n) \\ & \gamma_n \frac{n^*}{N_0} \left( 1 + \frac{2\gamma_n}{(2 - \gamma_n) (1 - K(c^*, n^*)) \varepsilon} + 2\gamma_n \right) < 2\varepsilon + 4\gamma_n \varepsilon. \end{aligned}$$

Substituting  $\lim_{N_0 \rightarrow +\infty} n^*/N_0$  from equation (6.29) yields:

$$\begin{aligned} & \gamma_n \left( 1 + \frac{2\gamma_n}{(2 - \gamma_n) (1 - K(c^*, n^*)) \varepsilon} + 2\gamma_n \right) < \varepsilon (2 + 4\gamma_n) \sqrt{1 + 2\varepsilon + \sqrt{1 + 2\varepsilon}} \\ & \varepsilon (1 + 2\gamma_n) + \frac{2\gamma_n}{(2 - \gamma_n) (1 - K(c^*, n^*))} < \frac{(2 + 4\gamma_n)}{\gamma_n} \varepsilon^2 \sqrt{1 + 2\varepsilon + \sqrt{1 + 2\varepsilon}} \end{aligned}$$

$$\begin{aligned}
\varepsilon + \frac{2\gamma_n}{(2 - \gamma_n)(1 - K(c^*, n^*))(1 + 2\gamma_n)} &< \frac{(2 + 4\gamma_n)}{\gamma_n(1 + 2\gamma_n)} \varepsilon^2 \sqrt{1 + 2\varepsilon + \sqrt{1 + 2\varepsilon}} \\
\varepsilon + \frac{2\gamma_n}{(2 - \gamma_n)(1 - K(c^*, n^*))(1 + 2\gamma_n)} &< \frac{2}{\gamma_n} \varepsilon^2 + 2\varepsilon + \sqrt{1 + 2\varepsilon + \sqrt{1 + 2\varepsilon}} \\
\frac{\varepsilon\gamma_n}{2} + \frac{\gamma_n^2}{(2 - \gamma_n)(1 - K(c^*, n^*))(1 + 2\gamma_n)} &< \varepsilon^2 \sqrt{1 + 2\varepsilon + \sqrt{1 + 2\varepsilon}}.
\end{aligned}$$

The expression for  $K$  given by equation (A.20) is decreasing in  $\gamma_n$ . Hence, the left-hand side of this inequality is increasing in  $\gamma_n$ . The right-hand side of the inequality is increasing in  $\varepsilon$ . If this inequality holds (that is, if  $\gamma_n$  is small relative to  $\varepsilon$ ), then the equilibrium firm profit is decreasing in the inefficiency coefficient.

### Proof of Lemma 8

Substituting the bargaining power function into the wage rate,  $\beta$ , results in:

$$\begin{aligned}
\beta &= K(c, n) \left( 1 - \varepsilon \frac{n(n+1)}{N_0(N_0+1)} \right) \\
&= \frac{n^2}{c + n^2} \left( \frac{N_0(N_0+1) - \varepsilon n(n+1)}{N_0(N_0+1)} \right).
\end{aligned}$$

Substituting the value of  $c = c^*$  (from equation 7.2), this equation becomes:

$$\beta = \frac{n^2}{n\sqrt{\frac{N_0(N_0+1) - \varepsilon n(n+1)}{2}}} \left( \frac{N_0(N_0+1) - \varepsilon n(n+1)}{N_0(N_0+1)} \right).$$

After some algebra, this becomes:

$$\beta = \sqrt{2} \frac{n\sqrt{N_0(N_0+1) - \varepsilon n(n+1)}}{N_0(N_0+1)}.$$

### Proof of Lemma 9

Taking the derivative of equation (7.2) with respect to the size of the union,  $n$ , (to calculate the marginal wage rate with respect to the number

of union members) yields:

$$\begin{aligned}
\frac{\partial \beta}{\partial n} &= \frac{n^2}{c+n^2} \left( -\varepsilon \frac{2n+1}{N_0(N_0+1)} \right) + \frac{2nc}{(c+n^2)^2} \left( 1 - \varepsilon \frac{n(n+1)}{N_0(N_0+1)} \right) \\
&= \frac{-\varepsilon n^2(2n+1)(c+n^2)}{N_0(N_0+1)(c+n^2)^2} + \frac{2nc[N_0(N_0+1) - \varepsilon n(n+1)]}{N_0(N_0+1)(c+n^2)^2} \\
&= \frac{2ncN_0(N_0+1) - \varepsilon n^2[(2n+1)(c+n^2) + 2c(n+1)]}{N_0(N_0+1)(c+n^2)^2}.
\end{aligned}$$

This marginal rate is non-negative if and only if:

$$2cN_0(N_0+1) \geq \varepsilon n[(2n+1)(c+n^2) + 2c(n+1)]$$

Equivalently:

$$\frac{2cN_0(N_0+1)}{\varepsilon} \geq n(2n^3 + n^2 + 4cn + 3c).$$

This condition and its unique positive solution  $n^{BR}$  are the same as in Proposition 14.

### **Proof of Corollary 13**

The result is implied by Corollary 12, which established when  $d\alpha^*/d\varepsilon < 0$ . This is because  $\beta$  is an increasing transformation of  $\alpha$  (for details, see the proof of Corollary 6).

# Appendix B

## Endogenous Union Inefficiency Model

### B.1 Model Setup

This appendix extends the Constant Wage Model of Chapter 3 by endogenizing the union inefficiency coefficient. This more realistic model is complicated to solve analytically because the number of endogenous variables is large. For this reason, this appendix focuses only on the key properties of the model and its equilibrium conditions. As in Chapter 3, output is unobservable or unverifiable and assumptions (3.1) through (3.5) hold. The only difference from Chapter 3 is that the union chooses the inefficiency coefficient in addition to the union wage. In particular, suppose union members always incur some minimal level of inefficiency, denoted  $\varepsilon_0 > 0$ , which we call the *fixed inefficiency*. The fixed inefficiency does not give union members any additional benefit, and this parameter is not under the control of any player.

Suppose the union can increase the inefficiency to higher levels by choosing the *variable inefficiency*, denoted  $\varepsilon_1$  (it can also decrease the inefficiency if it wishes). Assume that the inefficiency is the sum  $\varepsilon = \varepsilon_0 + \varepsilon_1$ , and  $0 \leq \varepsilon \leq 1$ . This means that the union can select any level  $\varepsilon_1$  for the variable

inefficiency such that  $0 \leq \varepsilon_1 \leq 1 - \varepsilon_0$ .<sup>1</sup>

The utility function of union members incorporates better working conditions (only available to union members) that is increasing in the variable inefficiency,  $\varepsilon_1$ . This may include, for example, freedom to take protected industrial action, better leave entitlements, fewer hours worked, adjustments to the flexibility of hours worked or the pace of work, smaller production targets, increased protection from job losses, increased training, or some form of on-work leisure. Let  $\lambda > 0$  be a constant measuring the intensity of the utility for the better working conditions. Let  $L : [0, 1 - \varepsilon_0] \rightarrow \mathbb{R}$ , defined at every  $\varepsilon_1 \in [0, 1 - \varepsilon_0]$  by  $L(\varepsilon_1) = \lambda\sqrt{\varepsilon_1}$ , represent the extra benefit for each union member of the better working conditions when variable inefficiency is  $\varepsilon_1$ . The term  $L(\varepsilon_1)$  is added to the utility of union members.

There will be two wage levels in equilibrium. Union members earn wage  $\alpha$  and non-members earn  $\alpha + L(\varepsilon_1)$ . The term  $L(\varepsilon_1)$  is a compensation premium that non-members obtain because they do not benefit from the same (relatively more pleasant) working conditions as union members do. Workers not in the union do not have incentives to join the union because if they do, their utility remains the same; they lose the premium on their wage, but benefit from better working conditions. The firm is willing to pay the compensation premium to non-unionized workers because of the threat effect (as described in Chapter 3).

In this model, the union anticipates for each variable inefficiency,  $\varepsilon_1$ , and union wage level,  $\alpha$ , the equilibrium number of workers,  $n^*$ . The proposed strategy profile is the same as in Chapter 3; workers with skills  $j \in \{1, \dots, n\}$  are union members, and workers with skills  $j > n$  are not in the union. As in Chapter 3, the workers' surplus is  $Y_w = G_U + G_N$ , where the shares to union members and non-members are, respectively:

$$G_U = \alpha n, \tag{B.1}$$

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<sup>1</sup>We use a different notation for each inefficiency type in order to make Appendix B and the proofs easier to follow, and also because it is intuitive to think of the inefficiency to be fixed or variable.

$$G_N = [\alpha + L(\varepsilon_1)](N_0 - n). \quad (\text{B.2})$$

## B.2 Equilibrium

The focus will be on an equilibrium where union workers receive the wage  $\alpha$  and non-union workers earn  $\alpha + L(\varepsilon_1)$ . The first result derives an expression for the union wage by combining equations (3.8), (B.1) and (B.2).

**Lemma 10** *The wage of a unionized worker, in terms of the workers' bargaining power, inefficiency coefficients, union size and working conditions parameter, is given by:*

$$\alpha = K(n) \frac{[N_0(N_0 + 1) - (\varepsilon_0 + \varepsilon_1)n(n + 1)]}{2N_0} - \frac{\lambda\sqrt{\varepsilon_1}(N_0 - n)}{N_0}.$$

The problem of the union is to maximize the utility of its members by choosing their salary,  $\alpha$ , and the variable inefficiency,  $\varepsilon_1$ , subject to budget balance. Mathematically:

$$\max_{\alpha \geq 0, 0 \leq \varepsilon_1 \leq 1} \alpha + L(\varepsilon_1) \quad \text{subject to equations (3.8), (B.1) and (B.2).}$$

Define the auxiliary variable  $\psi$  by:

$$\psi = \left( \frac{\lambda}{(n + 1)K(n)} \right)^2 [n(n + 1)K'(n) - K(n)].$$

In the case where the equilibrium variable inefficiency is zero (when  $\varepsilon_1^* = 0$  and, hence, the right hand side of equation B.3 is zero,  $\psi = 0$  and  $\varepsilon = \varepsilon_0$ ) the formula for the equilibrium union size (given by equation B.4 of the following proposition) reverts back to equation (3.11) of Chapter 3. This clearly occurs when the working conditions parameter is zero,  $\lambda = 0$ . In this sense,  $\psi$  is the difference between the models. Let  $\varepsilon_1^*$  be the variable inefficiency that solves the union's problem. The next proposition outlines the conditions for the equilibrium.

**Proposition 17** *Suppose assumptions (3.1) through (3.5) are satisfied. In addition, suppose that the elasticity of the bargaining function is not too large relative to the union size, precisely,  $\gamma < (2n - 1)n/(n + 1)$ . Then, there exists variable inefficiency,  $\varepsilon_1^*$ , that solves the union's problem. This inefficiency is given by:*

$$\varepsilon_1^* = \left( \frac{\lambda}{(n^* + 1)K(n^*)} \right)^2. \quad (\text{B.3})$$

To obtain  $\varepsilon_1^* < 1 - \varepsilon_0$ , there must be sufficiently many union members in the sense that  $\lambda/\sqrt{1 - \varepsilon_0} < (n + 1)K(n)$ . Assuming this condition holds, the union size  $n^*$  that solves the union's problem is either the upper corner solution,  $n^* = N_0$ , or an interior solution implicitly defined by:

$$\frac{K'(n^*)}{K(n^*)} = \frac{\varepsilon_0(2n^* + 1) + \psi/K(n^*)}{N_0(N_0 + 1) - \varepsilon_0 n^*(n^* + 1)}. \quad (\text{B.4})$$

Or equivalently:

$$\frac{K'(n^*)}{K(n^*)} = \frac{\varepsilon_0(2n^* + 1) + \varepsilon_1^*[n^*(n^* + 1)K'(n^*) - K(n^*)]/K(n^*)}{N_0(N_0 + 1) - \varepsilon_0 n^*(n^* + 1)}.$$

The equilibrium number of union members depends on the bargaining function,  $K(n)$ , and on the parameters for the workforce size, the exogenous inefficiency and working conditions. If  $\lambda = 0$ , then  $\psi = 0$  and equations (3.11) and (B.4), which implicitly define the equilibrium size of the union, become equivalent, as  $\varepsilon = \varepsilon_0$ . In this sense, the model is a generalization of the model with the exogenous inefficiency only. If  $\varepsilon_0 = 0$ , then the model is one of variable inefficiency only.

Proposition 17 shows that the equilibrium variable inefficiency depends on the working conditions parameter,  $\lambda$ , the equilibrium size of the union,  $n^*$ , and the workers' bargaining power,  $K(n)$ .

The model so far has considered a general bargaining function,  $K(n)$ . The next subsection restricts the bargaining function to increase the tractabil-

ity of the model. This allows us to obtain more insight into the equilibrium variables.

### B.3 Bargaining with Constant Elasticity

This subsection studies the endogenous inefficiency model under an assumption that the elasticity of the bargaining function, with respect to union size, is constant. Without this restriction, the model would be cumbersome to solve analytically. In this sense, the endogenous inefficiency model presented here is not so much a generalization of the exogenous inefficiency model (which restricted the inefficiency), but a different restriction.

Suppose throughout this subsection that the bargaining function is given by  $K(n) = n^\gamma/N_0^\gamma$ , for all  $n \in [0, N_0]$ , and some constant  $0 < \gamma < 1$ . The parameter  $\gamma$  represents the bargaining elasticity. Then,  $K'(n)/K(n) = \gamma/n$ . The next proposition outlines the equilibrium condition.

**Proposition 18** *Suppose the bargaining function is given by  $K(n) = n^\gamma/N_0^\gamma$ . The equilibrium union size,  $n^*$ , is either the upper corner solution  $n^* = N_0$  (all workers are unionized) or it is the interior solution which is implicitly defined by the first order condition:*

$$\gamma N_0(N_0 + 1) = \varepsilon_0 n^*[(2 + \gamma)n^* + (1 + \gamma)] + \lambda^2 N_0^{2\gamma} \frac{\gamma n^* + \gamma - 1}{n^{*2\gamma-1}(n^* + 1)^2}. \quad (\text{B.5})$$

Proposition 18 shows that the equilibrium level of union membership depends on the workforce size,  $N_0$ , exogenous inefficiency,  $\varepsilon_0$ , elasticity,  $\gamma$ , and the desire for improved working conditions,  $\lambda$ .

If  $\gamma = 1/2$ , equation (B.5) becomes a polynomial of degree four in the variable  $n$ . If  $\gamma$  is generic, but  $\lambda = 0$ , then it becomes a polynomial of degree two in variable  $n$ , and its unique positive solution  $n^*$  is the same as the one obtained in equation (3.14) when  $\varepsilon = \varepsilon_0$ .

Even with this restriction on the bargaining function it is cumbersome to solve analytically, so we restrict our attention to the case when the workforce



size,  $N_0$ , is large. The next result provides sufficient conditions for an interior solution to the union's problem; that is, for when  $n^* < N_0$ .

**Proposition 19** *Suppose the bargaining function is given by  $K(n) = n^\gamma / N_0^\gamma$  and that the workforce size grows large without bound,  $N_0 \rightarrow +\infty$ . Then, there exists an interior solution ( $n^* < N_0$ ) for the union's problem if and only if the inefficiency coefficient or desire for improved working conditions are sufficiently large. Mathematically, an interior solution exists if:*

$$\lim_{N_0 \rightarrow +\infty} \lambda > N_0 \sqrt{1 - \varepsilon_0 \left(1 + \frac{2}{\gamma}\right)}$$

or

$$\lim_{N_0 \rightarrow +\infty} \varepsilon_0 > \frac{\gamma}{2 + \gamma}.$$

If there is an interior solution, then the limit of the union density, as the size of the workforce grows large,  $N_0 \rightarrow +\infty$ , is given by:

$$\lim_{N_0 \rightarrow +\infty} \frac{n^*}{N_0} = \sqrt{\frac{\gamma}{\varepsilon_0(2 + \gamma)}}. \quad (\text{B.6})$$

Proposition 19 proves that all workers will be unionized if the inefficiency coefficient is sufficiently small. This is consistent with the result of Proposition 1. The intuition is that workers are better off in the union when there is no inefficiency, because it increases their bargaining power. It is only when the inefficiency is sufficiently large that it becomes profitable for workers to leave the union.

The result of Proposition 19 requires the fixed inefficiency to be positive,  $\varepsilon_0 > 0$ , because  $\varepsilon_0$  appears in the denominator of equation (B.6).

Proposition 19 also shows that the desire for good working conditions,  $\lambda$ , must be sufficiently large for some workers not to unionize. This is because the variable inefficiency,  $\varepsilon_1$ , is increasing in the working conditions parameter  $\lambda$ , and a higher variable inefficiency reduces the size of the surplus.

Proposition 19 also shows that the equilibrium union density is decreas-

ing in the fixed inefficiency; a result that is consistent with Corollary 2. Corollary 2 explains the intuition for this result.

The next result studies the equilibrium variable inefficiency and wages when the workforce size grows large.

**Proposition 20** *Suppose the bargaining function is given by  $K(n) = n^\gamma / N_0^\gamma$ . As the workforce size grows large, the equilibrium variable inefficiency falls to zero and the union and non-union wages converge. Mathematically:*

$$\lim_{N_0 \rightarrow +\infty} \varepsilon_1^* = 0,$$

$$\lim_{N_0 \rightarrow +\infty} \left( \alpha^* + \lambda \sqrt{\varepsilon_1^*} \right) = \lim_{N_0 \rightarrow +\infty} \alpha^*.$$

The results of Proposition 20 partly reflect the assumption on the skill distribution. In particular, as the workforce size grows, so too does the skill of the average worker and, hence, their level of production. With a higher average skill in the union, it makes it increasingly costly to provide improved working conditions to workers, so the union reduces the variable inefficiency to zero as the workforce grows infinitely large. Proposition 20 also shows that equilibrium wages of union and non-union workers converge as the workforce grows large. This is because the equilibrium variable inefficiency declines to zero and, hence, non-union workers do not need a compensation premium for differences in working conditions. Given the result of Proposition 20 (that is, when the workforce size grows large, the variable inefficiency is zero and the union and non-union wages converge), the model studied in this subsection is largely the same as the Constant Wage Model of Chapter 3, at least when the workforce size grows large.

## B.4 Summary

This appendix studied an extension to Chapter 3 that endogenized the union inefficiency coefficient. As in Chapter 3, output was unobservable and the

focus was on the properties of an equilibrium where lower skilled workers joined the labor union while higher skilled workers did not. The model was solved to find properties of the equilibrium variables. This appendix also studied the model under a restriction on the bargaining function. This improved the tractability of the model to some degree. A key finding under this restriction was that, as the workforce size grows large, the variable inefficiency converged to zero and the union and non-union wages converge.

## B.5 Proofs

### Proof of Lemma 10

Combining equations (3.8) and (B.1) with equation (B.2) results in:

$$\alpha n + (\alpha + L(\varepsilon_1))(N_0 - n) = G_U + G_N = Y_w = \frac{K(n)}{2} [N_0(N_0 + 1) - \varepsilon n(n + 1)].$$

Because  $\varepsilon = \varepsilon_0 + \varepsilon_1$  and  $L(\varepsilon_1) = \lambda\sqrt{\varepsilon_1}$ , then:

$$\alpha N_0 + \lambda\sqrt{\varepsilon_1}(N_0 - n) = \frac{K(n)}{2} [N_0(N_0 + 1) - (\varepsilon_0 + \varepsilon_1)n(n + 1)].$$

Equivalently:

$$2N_0(\alpha + \lambda\sqrt{\varepsilon_1}) = 2\lambda\sqrt{\varepsilon_1}n - \varepsilon_1 n(n + 1)K(n) + N_0(N_0 + 1)K(n) - \varepsilon_0 n(n + 1)K(n). \quad (\text{B.7})$$

Solving for  $\alpha$  completes the proof.

### Proof of Proposition 17

The equilibrium variable inefficiency and equation implicitly defining the size of the union are calculated as follows. Because of equation (B.7) and because  $2N_0 > 0$ , to maximize  $\alpha + L(\varepsilon_1)$ , it suffices to maximize the right-hand side of equation (B.7). Then:

$$\arg \max_{\alpha \geq 0, 0 \leq \varepsilon_1 \leq 1} \alpha + L(\varepsilon_1) = \arg \max_{\alpha \geq 0, 0 \leq \varepsilon_1 \leq 1} \alpha + \lambda\sqrt{\varepsilon_1}$$

$$= \arg \max_{0 \leq n \leq N_0, 0 \leq \varepsilon_1 \leq 1} 2\lambda\sqrt{\varepsilon_1}n - (\varepsilon_0 + \varepsilon_1)n(n+1)K(n) + N_0(N_0+1)K(n).$$

The partial derivatives of this equation with respect to  $n$  and  $\varepsilon_1$ , set to zero, are equivalent to (respectively):

$$2\lambda\sqrt{\varepsilon_1} + N_0(N_0+1)K'(n) = (\varepsilon_0 + \varepsilon_1)[(2n+1)K(n) + n(n+1)K'(n)], \quad (\text{B.8})$$

$$\frac{\lambda}{\sqrt{\varepsilon_1}} = (n+1)K(n). \quad (\text{B.9})$$

Let  $\varepsilon_1^*$  be the solution of this last equation. Then:

$$\varepsilon_1^* = \frac{\lambda^2}{[(n+1)K(n)]^2}.$$

To obtain  $\varepsilon_1^* < 1 - \varepsilon_0$ , there must be sufficiently many union members in the sense that  $\lambda/\sqrt{1 - \varepsilon_0} < (n+1)K(n)$ . Assuming this condition holds, and substituting  $\varepsilon_1^*$  into equation (B.8) leads to:

$$\begin{aligned} & \frac{2\lambda^2}{(n+1)K(n)} + N_0(N_0+1)K'(n) \\ &= \frac{\lambda^2[(2n+1)K(n) + n(n+1)K'(n)]}{[(n+1)K(n)]^2} + \varepsilon_0(2n+1)K(n) + \varepsilon_0n(n+1)K'(n). \end{aligned} \quad (\text{B.10})$$

Define the auxiliary variable  $\psi$  by:

$$\begin{aligned} \psi &= \frac{\lambda^2[(2n+1)K(n) + n(n+1)K'(n)]}{[(n+1)K(n)]^2} - \frac{2\lambda^2}{(n+1)K(n)} \\ &= \frac{\lambda^2}{[(n+1)K(n)]^2} [n(n+1)K'(n) - K(n)]. \end{aligned}$$

Substituting this auxiliary variable into equation (B.10) yields:

$$N_0(N_0+1)K'(n) = \varepsilon_0(2n+1)K(n) + \varepsilon_0n(n+1)K'(n) + \psi.$$

Dividing both sides by  $K(n)[N_0(N_0+1) - \varepsilon_0n(n+1)]$  and after some algebra,

this equality becomes:

$$N_0(N_0 + 1) \frac{K'(n)}{K(n)} = \varepsilon_0(2n + 1) + \varepsilon_0 n(n + 1) \frac{K'(n)}{K(n)} + \frac{\psi}{K(n)}.$$

Therefore:

$$\frac{K'(n)}{K(n)} = \frac{\varepsilon_0(2n + 1) + \psi/K(n)}{N_0(N_0 + 1) - \varepsilon_0 n(n + 1)}.$$

This equation implicitly defines the equilibrium size of the union which solves the union's problem. To see the solution is indeed a maximum of the union's problem, consider the following second order partial derivative test. The first order partial derivatives of the union wage with respect to  $n$  and  $\varepsilon_1$  were calculated previously and are given by equations (B.8) and (B.9), respectively. The second order partial derivative with respect to  $n$  is:

$$\begin{aligned} \frac{\partial^2 [\alpha + L(\varepsilon_1)]}{\partial n^2} &= N_0(N_0 + 1)K''(n) \\ &- (\varepsilon_0 + \varepsilon_1) [(2n + 1)K'(n) + 2K(n) + n(n + 1)K''(n) + (2n + 1)K'(n)]. \end{aligned}$$

Simplifying yields:

$$\frac{\partial^2 [\alpha + L(\varepsilon_1)]}{\partial n^2} = xK''(n) - (\varepsilon_0 + \varepsilon_1)[2(2n + 1)K'(n) + 2K(n) + n(n + 1)K''(n)].$$

This equation is negative because, by assumptions (3.1) through (3.5),  $K''(n) < 0$ ;  $K'(n) > 0$ ;  $K(n) > 0$ , which means that all terms in this equation are negative except for  $-(\varepsilon_0 + \varepsilon_1)n(n + 1)K''(n)$ ; however the term  $N_0(N_0 + 1)K''(n)$  (which is negative) more than offsets that term because  $N_0 \geq n$ . The other second order partial derivative, and the cross derivative, are, respectively:

$$\frac{\partial^2 [\alpha + L(\varepsilon_1)]}{\partial \varepsilon_1^2} = \frac{-\lambda}{2\varepsilon_1^{1.5}}$$

and

$$\frac{\partial^2 [\alpha + L(\varepsilon_1)]}{\partial n \partial \varepsilon_1} = -K(n) - (n + 1)K'(n).$$

For the solution to maximize the union's problem we require:

$$\frac{\partial^2 [\alpha + L(\varepsilon_1)]}{\partial n^2} \frac{\partial^2 [\alpha + L(\varepsilon_1)]}{\partial \varepsilon_1^2} - \left( \frac{\partial^2 [\alpha + L(\varepsilon_1)]}{\partial n \partial \varepsilon_1} \right)^2 > 0,$$

and

$$\frac{\partial^2 [\alpha + L(\varepsilon_1)]}{\partial n^2} < 0, \quad \text{or} \quad \frac{\partial^2 [\alpha + L(\varepsilon_1)]}{\partial \varepsilon_1^2} < 0.$$

Note that both  $\partial^2 [\alpha + L(\varepsilon_1)] / \partial n^2 < 0$  and  $\partial^2 [\alpha + L(\varepsilon_1)] / \partial \varepsilon_1^2 < 0$  (as described earlier in the proof) so the second requirement is satisfied. As regards the first requirement, the first term in the requirement will be positive because both  $\partial^2 [\alpha + L(\varepsilon_1)] / \partial n^2$  and  $\partial^2 [\alpha + L(\varepsilon_1)] / \partial \varepsilon_1^2$  are negative. The second term is a negative number squared, which is positive. Therefore, for the first requirement to be met the following condition must hold:

$$\frac{\partial [\alpha + L(\varepsilon_1)]^2}{\partial n^2} \frac{\partial [\alpha + L(\varepsilon_1)]^2}{\partial \varepsilon_1^2} > \left( \frac{\partial^2 [\alpha + L(\varepsilon_1)]}{\partial n \partial \varepsilon_1} \right)^2.$$

Equivalently:

$$\begin{aligned} [xK''(n) - (\varepsilon_0 + \varepsilon_1) (2(2n+1)K'(n) + 2K(n) + n(n+1)K''(n))] \frac{-\lambda}{2\varepsilon_1^{1.5}} \\ > [-K(n) - (n+1)K'(n)]^2. \end{aligned}$$

Substituting equation (B.9) for  $\lambda$  into this inequality and re-arranging yields:

$$\begin{aligned} - \frac{(n+1)K(n)}{2} [xK''(n) - (\varepsilon_0 + \varepsilon_1)] \\ - \frac{(n+1)K(n)}{2} [(2(2n+1)K'(n) + 2K(n) + n(n+1)K''(n))] \\ > \varepsilon_1 [-K(n) - (n+1)K'(n)]^2. \end{aligned}$$

On the left-hand side of this inequality, the terms with  $K''(n)$  are a net positive (that is,  $-xK''(n) > (\varepsilon_0 + \varepsilon_1)n(n+1)K''(n)$  because  $K''(n) < 0$  by

assumption 3.5). Using this fact, the above inequality will be true if:

$$\begin{aligned} \frac{(n+1)K(n)(\varepsilon_0 + \varepsilon_1)}{2} (2(2n+1)K'(n) + 2K(n)) \\ > \varepsilon_1 [-K(n) - (n+1)K'(n)]^2. \end{aligned}$$

After a little simplification on the left-hand side of the inequality and expanding the right-hand side we obtain:

$$\begin{aligned} (n+1)K(n)(\varepsilon_0 + \varepsilon_1) [(2n+1)K'(n) + K(n)] \\ > \varepsilon_1 [K(n)^2 + 2K(n)K'(n)(n+1) + (n+1)^2K'(n)^2]. \end{aligned}$$

Dividing both sides of this inequality by  $K(n)(n+1)(\varepsilon_0 + \varepsilon_1)$  yields:

$$(2n+1)K'(n) + K(n) > \frac{\varepsilon_1}{(\varepsilon_0 + \varepsilon_1)} \left[ \frac{K(n)}{(n+1)} + 2K'(n) + (n+1) \frac{K'(n)^2}{K(n)} \right].$$

Because  $\varepsilon_1/((\varepsilon_0 + \varepsilon_1)) < 1$ , this inequality holds if:

$$(2n+1)K'(n) + K(n) > \frac{K(n)}{(n+1)} + 2K'(n) + (n+1) \frac{K'(n)^2}{K(n)}.$$

Consolidating the  $K'(n)$  and  $K(n)$  terms yields:

$$K'(n)(2n-1) + K(n) \left( 1 - \frac{1}{n+1} \right) > (n+1) \frac{K'(n)^2}{K(n)}.$$

Multiplying both sides of the inequality by  $K(n)$  and consolidating the  $K'(n)$  and  $K(n)$  terms yields:

$$K'(n) [(2n-1)K(n) - (n+1)K'(n)] + K(n)^2 \left( 1 - \frac{1}{n+1} \right) > 0.$$

The term  $K(n)^2 (1 - 1/(n+1)) > 0$  is clearly positive. Because of this, and because  $K'(n) > 0$  by assumption (3.4), the above inequality will be satisfied

if:

$$(2n - 1)K(n) > (n + 1)K'(n).$$

Equivalently:

$$\begin{aligned}\frac{2n - 1}{n + 1} &> \frac{K'(n)}{K(n)} \\ \frac{2n - 1}{n + 1}n &> \frac{K'(n)}{K(n)}n \\ \frac{2n - 1}{n + 1}n &> \gamma.\end{aligned}$$

This inequality is assumed to hold at the beginning of Proposition 17.

**Proof of Proposition 18**

Suppose that  $K(n) = n^\gamma/N_0^\gamma$ , for all  $n \in [0, N_0]$ , and some constant  $0 < \gamma < 1$ . Then,  $K'(n)/K(n) = \gamma/n$ , and equation (B.4) becomes:

$$\gamma N_0(N_0 + 1) - \gamma \varepsilon_0 n(n + 1) = \varepsilon_0 n(2n + 1) + \frac{n\psi}{K(n)}.$$

Substituting  $\psi$  we obtain:

$$\gamma N_0(N_0 + 1) - \gamma \varepsilon_0 n(n + 1) = \varepsilon_0 n(2n + 1) + \frac{n\lambda^2 [n(n + 1)K'(n) - K(n)]}{(n + 1)^2 K(n)^3}.$$

Substituting the formulas for  $K(n)$  and  $K'(n)$ , and after some algebra, this becomes:

$$\gamma N_0(N_0 + 1) = \varepsilon_0 n[(2 + \gamma)n + (1 + \gamma)] + \lambda^2 N_0^{2\gamma} \frac{\gamma n + \gamma - 1}{n^{2\gamma-1}(n + 1)^2}.$$

**Proof of Proposition 19**

*I. Condition for Interior Solution in terms of the Fixed Inefficiency*



Re-writing equation (B.5) yields:

$$\begin{aligned} \varepsilon_0 [n^{2\gamma+4}(2+\gamma) + n^{2\gamma+3}(3+\gamma) - n^{2\gamma+2}\gamma - n^{2\gamma+1}(1+\gamma)] \\ - n\lambda^2 N_0^{2\gamma}(1-\gamma) + n^2\gamma\lambda^2 N_0^{2\gamma} \\ = n^{2\gamma+2}\gamma x + n^{2\gamma+1}2\gamma x + n^{2\gamma}\gamma x. \end{aligned}$$

We are interested in the case where the exogenous inefficiency is sufficiently large, so consider:

$$\begin{aligned} \varepsilon_0 [N_0^{2\gamma+4}(2+\gamma) + N_0^{2\gamma+3}(3+\gamma) - N_0^{2\gamma+2}\gamma - N_0^{2\gamma+1}(1+\gamma)] \\ - N_0\lambda^2 N_0^{2\gamma}(1-\gamma) + N_0^2\gamma\lambda^2 N_0^{2\gamma} \\ > N_0^{2\gamma+2}\gamma x + N_0^{2\gamma+1}2\gamma x + N_0^{2\gamma}\gamma x. \end{aligned}$$

Solving for  $\varepsilon_0$  and simplifying yields:

$$\varepsilon_0 > \frac{N_0\gamma x + 2\gamma x + \gamma(N_0 + 1) + N_0 - \gamma\lambda^2 + \lambda^2(1-\gamma)}{N_0^3(2+\gamma) + N_0^2(3+\gamma) - N_0\gamma - (1+\gamma)}.$$

Taking the limit as  $N_0$  grows yields:

$$\lim_{N_0 \rightarrow +\infty} \varepsilon_0 > \frac{\gamma}{2+\gamma}.$$

## II. Condition for Interior Solution in terms of the Working Conditions Parameter

Re-writing equation (B.5) yields:

$$\begin{aligned} -n^{2\gamma+3}[\varepsilon_0(2+\gamma)] - n^{2\gamma+2}[\varepsilon_0(3+\gamma)] + n^{2\gamma+1}[\gamma(x+\varepsilon_0)] \\ + n^{2\gamma}[2\gamma x + \varepsilon_0(1+\gamma)] + n^{2\gamma-1}\gamma x - \lambda^2[n\gamma N_0^{2\gamma} - N_0^{2\gamma}(1-\gamma)] = 0. \end{aligned}$$

Because we are interested in the case where  $\lambda$  is sufficiently large, consider:

$$\begin{aligned} & -N_0^{2\gamma+3} [\varepsilon_0(2+\gamma)] - N_0^{2\gamma+2} [\varepsilon_0(3+\gamma)] + N_0^{2\gamma+1} [\gamma(x+\varepsilon_0)] \\ & + N_0^{2\gamma} [2\gamma x + \varepsilon_0(1+\gamma)] + N_0^{2\gamma-1} \gamma x < \lambda^2 [N_0 \gamma N_0^{2\gamma} - N_0^{2\gamma}(1-\gamma)]. \end{aligned}$$

Dividing by  $N_0^{2\gamma}$  and solving for  $\lambda$  yields:

$$\begin{aligned} & \frac{-N_0^2 [\varepsilon_0(2+\gamma)] - N_0 [\varepsilon_0(3+\gamma)]}{\gamma - \left(\frac{1-\gamma}{N_0}\right)} + \\ & + \frac{\gamma(x+\varepsilon_0) + 2\gamma(N_0+1) + \frac{\varepsilon_0(1+\gamma)}{N_0} - \gamma \left(1 + \frac{1}{N_0}\right)}{\gamma - \left(\frac{1-\gamma}{N_0}\right)} < \lambda^2. \end{aligned}$$

Taking the limit as  $N_0$  grows large yields:

$$\lim_{N_0 \rightarrow +\infty} \lambda^2 > \frac{N_0^2 [\gamma(1-\varepsilon_0) - 2\varepsilon_0] - N_0 [\gamma(2+\varepsilon_0) + 3\varepsilon_0]}{\gamma}.$$

Then:

$$\lim_{N_0 \rightarrow +\infty} \lambda > N_0 \sqrt{1 - \varepsilon_0 \left(1 + \frac{2}{\gamma}\right)}.$$

### III. Union Density

Re-writing equation (B.5) yields:

$$\begin{aligned} & [\gamma x - \varepsilon_0 n^2(2+\gamma) + \varepsilon_0 n(1+\gamma)] [n^{2\gamma+1} + 2n^{2\gamma} + n^{2\gamma-1}] \\ & = \lambda^2 N_0^{2\gamma} [\gamma n + \gamma - 1]. \end{aligned}$$

Expanding and simplifying the left-hand side of this equality allows for the left-hand side to be written as:

$$\begin{aligned} & \gamma x n^{2\gamma+1} - \varepsilon_0(2+\gamma)n^{2\gamma+3} + \varepsilon_0(1+\gamma)n^{2\gamma+2} + \gamma x 2n^{2\gamma} - \varepsilon_0(2+\gamma)2n^{2\gamma+2} \\ & + \varepsilon_0(1+\gamma)2n^{2\gamma+1} + \gamma x n^{2\gamma-1} - \varepsilon_0(2+\gamma)n^{2\gamma+1} + \varepsilon_0(1+\gamma)n^{2\gamma}. \end{aligned}$$

Substituting this expansion into the equality and simplifying both sides of the equality yields:

$$\begin{aligned}
& -n^{2\gamma+3}\varepsilon_0(2+\gamma) + n^{2\gamma+2}[\varepsilon_0(1+\gamma) - \varepsilon_0(2+\gamma)2] \\
& + n^{2\gamma+1}[\gamma x + \varepsilon_0(1+\gamma)2 - \varepsilon_0(2+\gamma)] + n^{2\gamma}[\gamma x 2 + \varepsilon_0(1+\gamma)] \\
& + \gamma x n^{2\gamma-1} = \lambda^2 N_0^{2\gamma}(\gamma n + \gamma - 1).
\end{aligned}$$

Dividing all terms by  $N_0^{2\gamma+4}$ , and after some algebra, we obtain:

$$\begin{aligned}
& n^{2\gamma+4} \left[ \frac{-\varepsilon_0(2+\gamma)}{N_0^{2\gamma+4}} \right] + n^{2\gamma+3} \left[ \frac{-\varepsilon_0(3+\gamma)}{N_0^{2\gamma+4}} \right] + n^{2\gamma+2} \left[ \frac{\gamma(x + \varepsilon_0)}{N_0^{2\gamma+4}} \right] \\
& + n^{2\gamma+1} \left[ \frac{2\gamma x + \varepsilon_0(1+\gamma)}{N_0^{2\gamma+4}} \right] + n^{2\gamma} \left[ \frac{\gamma x}{N_0^{2\gamma+4}} \right] + n^2 \left[ \frac{-\gamma \lambda^2 N_0^{2\gamma}}{N_0^{2\gamma+4}} \right] \\
& + n \left[ \frac{\lambda^2 N_0^{2\gamma}(1-\gamma)}{N_0^{2\gamma+4}} \right] = 0.
\end{aligned}$$

Taking the limit as  $N_0$  grows large yields:

$$\lim_{N_0 \rightarrow +\infty} n^{2\gamma+4} \left[ \frac{-\varepsilon_0(2+\gamma)}{N_0^{2\gamma+4}} \right] + n^{2\gamma+2} \left[ \frac{\gamma x}{N_0^{2\gamma+4}} \right] = 0.$$

Thus:

$$\lim_{N_0 \rightarrow +\infty} \frac{n}{N_0} = \sqrt{\frac{\gamma}{\varepsilon_0(2+\gamma)}}.$$

### **Proof of Proposition 20**

#### *I. Variable Inefficiency Falls to Zero as $N_0$ Grows*

Substituting  $K(n) = n^\gamma / N_0^\gamma$  into the equation for the variable inefficiency (given by equation B.3) yields:

$$\varepsilon_1^* = \left( \frac{\lambda}{(n^* + 1) \frac{n^\gamma}{N_0^\gamma}} \right)^2.$$

This clearly converges to zero as  $N_0$  grows large because (a) the elasticity and working condition parameter are constant, and (b) the result of Proposition

19 that the union density is constant as  $N_0$  grows large.

## II. The Union and Non-Union Wages Converge as $N_0$ Grows

The union wage is:

$$\alpha = \frac{K(n) [N_0(N_0 + 1) - (\varepsilon_0 + \varepsilon_1)n(n + 1)]}{2N_0} - \frac{\lambda\sqrt{\varepsilon_1}(N_0 - n)}{N_0}.$$

Taking the limit as the workforce size grows large,  $N_0 \rightarrow +\infty$ , in both sides of this equation yields:

$$\lim_{N_0 \rightarrow +\infty} \alpha = \frac{n^\gamma [N_0^2 - (\varepsilon_0 + \varepsilon_1)n^2]}{2N_0^{\gamma+1}}.$$

The non-union wage is:

$$\alpha + \lambda\sqrt{\varepsilon_1} = \frac{\frac{n^\gamma}{N_0^\gamma} [N_0(N_0 + 1) - (\varepsilon_0 + \varepsilon_1)n(n + 1)]}{2N_0} - \frac{\lambda\sqrt{\varepsilon_1}(N_0 - n)}{N_0} + \lambda\sqrt{\varepsilon_1}.$$

Taking the limit as the workforce size grows large,  $N_0 \rightarrow +\infty$ , in both sides of the equation results in:

$$\lim_{N_0 \rightarrow +\infty} (\alpha + \lambda\sqrt{\varepsilon_1}) = \frac{n^\gamma [N_0^2 - (\varepsilon_0 + \varepsilon_1)n^2]}{2N_0^{\gamma+1}}.$$

Clearly,  $\lim_{N_0 \rightarrow +\infty} (\alpha + \lambda\sqrt{\varepsilon_1}) = \lim_{N_0 \rightarrow +\infty} \alpha$ . This completes the proof.

# Appendix C

## A Note on Equilibrium Stability

This appendix studies the stability condition described in Chapter 6. For the purpose of this appendix, fix a particular Nash equilibrium,  $(c^*, n^*)$  such that  $c^* > 0$  and  $0 < n^* < N_0$ . All the equations and conditions of this appendix should be read as if they are evaluated at this Nash equilibrium. By an abuse of notation, we may not explicitly say this in the equations and conditions that follow.

The three possible situations that hold in equilibrium  $(c^*, n^*)$ , as described in Lemma 6, are:

$$\begin{aligned}\text{Case 1 : } K_{cn} &> \frac{-Y_n K_c}{Y} && \Leftrightarrow && \alpha_{nc} > 0, \pi_{cn} < 0 \\ \text{Case 2 : } K_{cn} &< \frac{-Y_n K_c}{Y} && \Leftrightarrow && \alpha_{nc} < 0, \pi_{cn} > 0 \\ \text{Case 3 : } K_{cn} &= \frac{-Y_n K_c}{Y} && \Leftrightarrow && \alpha_{nc} = 0, \pi_{cn} = 0.\end{aligned}$$

This appendix calculates the partial derivatives discussed in Section 6.3 and shows that the stability condition (also discussed in Section 6.3) holds in equilibrium. The stability condition requires that  $\alpha_{nn}\pi_{cc} - \alpha_{nc}\pi_{nc} > 0$ .

First, to see that  $\alpha_{nn} < 0$ , take the second order partial derivative of

equation (6.15) with respect to  $n$ , which yields:

$$\alpha_{nn} = \frac{1}{2N_0}(K_{nn}(N_0(N_0 + 1) - \varepsilon n(n + 1)) - 2K_n\varepsilon(2n + 1) - 2K\varepsilon).$$

This is negative because  $K_{nn} \leq 0$  (by assumption 6.6) and  $K_n > 0$  (by assumption 6.3). This proves that  $\alpha_{nn} < 0$ .

Second, to see that  $\pi_{cc} < 0$ , take the second order partial derivative of equation (6.12) with respect to  $c$ , which yields  $\pi_{cc} = -K_{cc}Y$ . This is negative because  $K_{cc} > 0$  (by assumption 6.7).

As  $\alpha_{nn} < 0$  and  $\pi_{cc} < 0$ , to achieve  $\alpha_{nn}\pi_{cc} - \alpha_{nc}\pi_{nc} > 0$  (the stability condition described in Section 6.3), it suffices to prove that the product of the cross partial derivatives must be zero or negative,  $\alpha_{nc}\pi_{nc} \leq 0$ . This is indeed the case in an equilibrium. We claim that:

**C.1** if  $K_{nc} > 0$ , then  $\alpha_{nc} > 0$  and  $\pi_{nc} < 0$ .

**C.2** if  $K_{nc} < 0$ , then:

**C.2.1**  $\alpha_{nc} > 0$  and  $\pi_{cn} < 0$  if Case 1 of Lemma 6 holds

**C.2.2**  $\alpha_{nc} < 0$  and  $\pi_{cn} > 0$  if Case 2 of Lemma 6 holds

**C.2.3**  $\alpha_{nc} = 0$  and  $\pi_{cn} = 0$  if Case 3 of Lemma 6 holds.

To prove the claims C.1 through C.2.3 hold, we need to calculate the partial cross derivatives  $\alpha_{nc}$  and  $\pi_{cn}$  first. These are:

$$\alpha_{nc} = \frac{1}{2N_0}(K_{nc}(N_0(N_0 + 1) - \varepsilon n(n + 1)) - K_c\varepsilon(2n + 1)) = \frac{K_{nc}Y + K_cY_n}{N_0}$$

$$\pi_{cn} = -K_{cn}Y + K_c\frac{\varepsilon(2n + 1)}{2} = -K_{cn}Y - K_cY_n.$$

Equivalently:

$$\pi_{cn} = -N_0\alpha_{nc}.$$

To see that C.1 holds, suppose  $K_{nc} > 0$ . In this case, it is clear that  $\alpha_{nc} > 0$  and  $\pi_{cn} < 0$  (using the fact that  $K_c < 0$  by assumption 6.5), and  $Y_n < 0$ .

To see that C.2.1 through C.2.3 hold, suppose  $K_{nc} < 0$ . In this case,  $\alpha_{nc} < 0$  and  $\pi_{cn} > 0$  if  $-K_{cn}Y > K_cY_n$ , or equivalently if  $K_{cn} < -K_cY_n/Y$ . This is the same inequality as in Case 2 of Lemma 6. (To see that cases C.2.1 and C.2.3 hold, simply change the relevant inequalities).

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